

SUPERPFAFFIAN

P. LAVAUD

ABSTRACT. Let $V = V_0 \oplus V_1$ be a real finite dimensional supervector space provided with a non-degenerate antisymmetric even bilinear form B . Let $\mathfrak{spo}(V)$ be the Lie superalgebra of endomorphisms of V which preserve B . We consider $\mathfrak{spo}(V)$ as a supermanifold. We show that a choice of an orientation of V_1 and of a square root i of -1 determines a very interesting generalized function on the supermanifold $\mathfrak{spo}(V)$, the *superPfaffian*.

When $V = V_1$, $\mathfrak{spo}(V)$ is the orthogonal Lie algebra $\mathfrak{so}(V_1)$ and the superPfaffian is the usual Pfaffian, a square root of the determinant.

When $V = V_0$, $\mathfrak{spo}(V)$ is the symplectic Lie algebra $\mathfrak{sp}(V_0)$ and the superPfaffian is a constant multiple of the Fourier transform of one the two minimal nilpotent orbits in the dual of the Lie algebra $\mathfrak{sp}(V_0)$, and is an analytic square root of the inverse of the determinant in the open subset of invertible elements of $\mathfrak{spo}(V)$.

Our opinion is that the superPfaffians (there are four of them, corresponding to the two orientations on V_1 , and to the two square roots of -1) are fundamental objects. At least, they occur in the study of equivariant cohomology of supermanifolds ([Lav98, Lav04]), and in the study of the metaplectic representation of the metaplectic group with Lie algebra $\mathfrak{spo}(V)$. In this article, we present the definition and some basic properties of the superPfaffians.

INTRODUCTION

Let V be an oriented finite dimensional real vector space provided with a non degenerate symmetric bilinear form B . On $\mathfrak{so}(V)$, the Pfaffian is a polynomial square root of the determinant, and it is well known that the Pfaffian of $X \in \mathfrak{so}(V)$ can be defined by a suitable Berezin integral (the notation $\int_V d_V$ for the integral is defined in section 1.4)

$$(1) \quad \int_V d_V(v) \exp\left(-\frac{1}{2}B(v, Xv)\right)$$

over the vector space V seen as an odd space (cf. for example [BGV92] and section 2 below).

This definition still have a formal meaning for a real supervector space $V = V_0 \oplus V_1$ provided with a non-degenerate antisymmetric even bilinear form B and an orientation of V_1 : V_0 is a symplectic vector space, V_1 is an oriented vector space provided with a non degenerate symmetric bilinear form, and this structure provides us with a well defined *Liouville integral* d_V on the supermanifold V (it is a specific normalization of the Berezin integral on V). However, the integral (1) is convergent only when X is in an open subset of $\mathfrak{spo}(V)$ (definition is given in section 2 below), where it is (as expected, and well known), a square root of the inverse Berezinian (Definition is given in section 1.9 below). We call this function the superPfaffian. For instance, if $V = V_0$, then for $X \in \mathfrak{sp}(V)$, $B(v, Xv)$ is a quadratic form on V , and the integral (1) is convergent when

it is positive definite. In this case, $\det(X)$ is strictly positive, and the superPfaffian is the positive square root of $1/\det(X)$.

Since the inverse Berezinian is not polynomial (it is only a rational function when V_0 is not 0), there is no natural extension of this superPfaffian to a function on the supermanifold $\mathfrak{spo}(V)$. The purpose of this article is to show that there is a natural extension of this superPfaffian as a *generalized function* on the supermanifold $\mathfrak{spo}(V)$. Notice that the superPfaffian is 0 when $\dim(V_1)$ is odd. Let $m = \dim(V_0)$ (which is even) and $n = \dim(V_1)$ (which we assume now to be even). Let $\mathbf{i} \in \mathbb{C}$ be a square root of -1 . We define the superPfaffian by the formula:

$$(2) \quad \mathbf{i}^{(m-n)/2} \int_V d_V(v) \exp\left(-\frac{\mathbf{i}}{2} B(v, Xv)\right).$$

Of course we prove that (2) has a well defined meaning as a generalized function of X on the supermanifold $\mathfrak{spo}(V)$.

The superPfaffian has very nice properties. As in [MQ86] for $V = V_1$, some of them follow from the transformations properties of d_V under linear or affine change of variables.

a) It is an analytic function in the open set where the inverse Berezinian is defined, and, in this open set, it is a square root of the inverse Berezinian.

b) In the open set where the integral (1) is convergent, it is equal to the function given by (1).

c) It is a boundary value of an holomorphic function defined in a specific cone of $\mathfrak{spo}(V \otimes \mathbb{C})$

d) It is harmonic. More precisely, let $SpO(V)$ be the supergroup of endomorphisms of V which preserve B . The superPfaffian is annihilated by the homogeneous constant coefficient differential operators on $\mathfrak{spo}(V)$ which are of degree > 0 and $SpO(V)$ -invariant.

We prove that properties a) and c) determine the superPfaffian up to sign.

For some values of m and n , we are able to prove that properties a) and d) determine the superPfaffian up to sign and complex conjugation. However, we do not know if this is true in general.

One main motivation for this study comes from differential supergeometry. The (equivariant) Euler form of a (equivariant) real Euclidean oriented fiber bundle is equal to the Pfaffian of the (equivariant) curvature of an (equivariant) connection on this bundle. With complications partly due to the fact that the superPfaffian is not a function but only a generalized function, this is still true in supergeometry (cf [Lav98, ?]). Thus the superPfaffian plays an important role in the equivariant cohomology of supermanifolds, in particular in relation with the localization formula. In fact, formulas (1) and (2) may be considered as particular typical cases of the localization formula.

A second motivation is the close relationship between the superPfaffian and the distribution character of the metaplectic representation of the simply connected Lie supergroup with Lie superalgebra $\mathfrak{spo}(V)$. This will be studied in another paper.

I wish to thank Michel Duflo who introduced me to supermathematics and spent much of his time to suggest to me many deep improvements to this paper.

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1. PREREQUISITES

1.1. Notations. In this article, unless otherwise specified, all supervector spaces and superalgebras will be real. If V is a supervector space, we denote by V_0 its even part and by V_1 its odd part. If v is a non zero homogeneous element of V , we denote by $p(v) \in \mathbb{Z}/2\mathbb{Z}$ its parity. We put $\dim(V) = (\dim(V_0), \dim(V_1))$. We denote by V^* the dual supervector space $\text{Hom}(V, \mathbb{R})$. If V and W are supervector spaces, $V \otimes W$ and $W \otimes V$ are supervector spaces, and they are identified using the rule of signs (for non zero homogenous $v \in V$ and $w \in W$ we identify $v \otimes w$ and $(-1)^{p(v)p(w)}w \otimes v$). We denote by $S(V)$ the symmetric algebra of V . Recall that it is equal to $S(V_0) \otimes \Lambda(V_1)$, where $S(V_0)$ and $\Lambda(V_1)$ are the classical symmetric and exterior algebras of the corresponding

ungraded vector spaces. We use the notation $\Lambda(U)$ only for ungraded vector spaces U . So, if V is a supervector space, $\Lambda(V)$ is the exterior algebra of the underlying vector space.

Let $(m, n) \in \mathbb{N} \times \mathbb{N}$. We denote by $\mathbb{R}^{(m, n)}$ the supervector space of dimension (m, n) such that $V_0 = \mathbb{R}^m$ and $V_1 = \mathbb{R}^n$.

We choose a square root \mathbf{i} of -1 .

1.2. Near superalgebras. We say that a commutative superalgebra \mathcal{P} is *near* if it is finite dimensional, local, and with \mathbb{R} as residual field. They are the *algèbres proches* of Weil [Wei53]. For $\alpha \in \mathcal{P}$, we denote by $\mathbf{b}(\alpha)$ the canonical projection of α in \mathbb{R} ($\mathbf{b}(\alpha)$ is the *body* of α , and $\alpha - \mathbf{b}(\alpha)$ —a nilpotent element of \mathcal{P} — the *soul* of α , according to the terminology of [DeW84]). Let $\alpha \in \mathcal{P}_0$ be an even element. Let $\phi \in \mathcal{C}^\infty(\mathbb{R}, W)$ be a smooth function defined in a neighborhood of $\mathbf{b}(\alpha)$ in \mathbb{R} , with values in some Fréchet supervector space W . We freely use the notation:

$$(3) \quad \phi(\alpha) = \sum_{k=0}^{\infty} \frac{(\alpha - \mathbf{b}(\alpha))^k}{k!} \phi^{(k)}(\mathbf{b}(\alpha)) \in W \otimes \mathcal{P}.$$

In particular, if $\alpha \in \mathcal{P}_0$ is invertible, its absolute value $|\alpha| \in \mathcal{P}_0$ is defined by the formula:

$$(4) \quad |\alpha| = \frac{|\mathbf{b}(\alpha)|}{\mathbf{b}(\alpha)} \alpha,$$

and if $\mathbf{b}(\alpha) > 0$, its square root is defined by the finite sum

$$\sqrt{\alpha} = \sqrt{\mathbf{b}(\alpha)} \left(1 + \frac{1}{2} \left(\frac{\alpha}{\mathbf{b}(\alpha)} - 1 \right) - \frac{1}{2^2 2!} \left(\frac{\alpha}{\mathbf{b}(\alpha)} - 1 \right)^2 + \frac{3}{2^3 3!} \left(\frac{\alpha}{\mathbf{b}(\alpha)} - 1 \right)^3 - \frac{3.5}{2^4 4!} \left(\frac{\alpha}{\mathbf{b}(\alpha)} - 1 \right)^4 + \dots \right),$$

where $\sqrt{\lambda}$ is the unique positive square root of $\lambda > 0$. A notation like $\sqrt{|\alpha|}$ (for $\alpha \in \mathcal{P}_0$ invertible) is not ambiguous, because if $f \in \mathcal{C}^\infty(\mathbb{R}^+)$ and $\mathbf{g} \in \mathcal{C}^\infty(\mathbb{R}, \mathbb{R}^+)$ (where $\mathbb{R}^+ = \{x \in \mathbb{R}, x \geq 0\}$), for $\alpha \in \mathcal{P}_0$ $f \circ g(\alpha) = f(g(\alpha))$.

1.3. Supermanifolds. By a *supermanifold* we mean a smooth real supermanifold as in [Kos77], [Ber87], [BL75]. Let V be a finite dimensional supervector space. We denote also by V the associated supermanifold. In this paper we mostly use this kind of supermanifolds, and we recall some relevant definitions in this particular case.

Let $\mathcal{U} \subset V_0$ be an open set. We put

$$\mathcal{C}_V^\infty(\mathcal{U}) = \mathcal{C}^\infty(\mathcal{U}) \otimes \Lambda(V_1^*),$$

where $\mathcal{C}^\infty(\mathcal{U})$ is the usual algebra of smooth real valued functions defined in \mathcal{U} , and $\Lambda(V_1^*)$ is the exterior algebra of V_1^* . We say that $\mathcal{C}_V^\infty(\mathcal{U})$ is the superalgebra of *smooth functions on V defined in \mathcal{U}* . The supermanifold V is by definition the topological space V_0 equipped with the sheaf of superalgebras \mathcal{C}_V^∞ . Notice that if \mathcal{U} is not empty, there is a canonical inclusion $S(V^*) \subset \mathcal{C}^\infty(\mathcal{U})$. The corresponding elements are called *polynomial functions*. One can also define rational functions. Similarly, if W is a Fréchet supervector space (for instance $W = \mathbb{C}$), we denote by $\mathcal{C}_V^\infty(\mathcal{U}, W) = \mathcal{C}^\infty(\mathcal{U}, W) \otimes \Lambda(V_1)^*$ the space of W -valued smooth functions.

Let \mathcal{P} be a near superalgebra. We put

$$V_{\mathcal{P}} = (V \otimes \mathcal{P})_0.$$

It is called the *set of points of V with values in \mathcal{P}* . Extending the body $\mathbf{b} : \mathcal{P} \rightarrow \mathbb{R}$ to a map $V \otimes \mathcal{P} \rightarrow V$, and restricting it to $V_{\mathcal{P}}$, we obtain a map, also called the body and denoted by \mathbf{b} ,

$$\mathbf{b} : V_{\mathcal{P}} \rightarrow V_0.$$

Let $\mathcal{U} \subset V_0$ be an open set. We denote by $V_{\mathcal{P}}(\mathcal{U}) \subset V_{\mathcal{P}}$ the inverse image of \mathcal{U} in $V_{\mathcal{P}}$. It is known that $V_{\mathcal{P}}(\mathcal{U})$ is canonically identified to the set of (even) algebra homomorphisms $\mathcal{C}_V^{\infty}(\mathcal{U}) \rightarrow \mathcal{P}$. Let $v \in V_{\mathcal{P}}(\mathcal{U})$. We will denote the corresponding character by $\phi \mapsto \phi(v)$, and say that $\phi(v) \in \mathcal{P}$ is the value of $\phi \in \mathcal{C}_V^{\infty}(\mathcal{U})$ at the point v .

For example, let $v = v_i p^i \in V_{\mathcal{P}}$ (we use Einstein's summation rule, and considering tensorisation by \mathcal{P} as an extension of scalars, we write $v_i p^i$ instead of $v_i \otimes p^i$) where the $(v_i, p^i) \in V \times \mathcal{P}$ are a finite number of pair of homogeneous elements with the same parity. Then

$$\mathbf{b}(v) = v_i \mathbf{b}(p^i),$$

which is in V_0 since $\mathbf{b}(p^i) = 0$ if p^i is odd. Let $\phi \in V^*$. We denote by the same letter the corresponding element in $\mathcal{C}_V^{\infty}(V_0)$. Then $\phi(v) = \phi(v_i) p^i$, and this formula in fact completely determines the bijection between $V_{\mathcal{P}}(\mathcal{U})$ and the set of even homomorphisms of algebras $\mathcal{C}_V^{\infty}(\mathcal{U}) \rightarrow \mathcal{P}$ (cf. [Wei53]).

For $\phi \in \mathcal{C}_V^{\infty}(\mathcal{U})$, we denote by $\phi_{\mathcal{P}}$ the corresponding function $v \mapsto \phi(v)$ defined in $V_{\mathcal{P}}(\mathcal{U})$. Then $\phi_{\mathcal{P}} \in \mathcal{C}^{\infty}(V_{\mathcal{P}}(\mathcal{U}), \mathcal{P})$. The importance of this construction is that for \mathcal{P} large enough (for instance if \mathcal{P} is an exterior algebra $\Lambda \mathbb{R}^N$ with $N \geq \dim(V_1)$), the map $\phi \mapsto \phi_{\mathcal{P}}$ is injective, which allows more or less to treat ϕ as an ordinary function.

We emphasize the special case $\mathcal{P} = \mathbb{R}$. Then $V_{\mathbb{R}} = V_0$, $V_{\mathbb{R}}(\mathcal{U}) = \mathcal{U}$, and $\phi_{\mathbb{R}}$ is the projection of $\phi \in \mathcal{C}_V^{\infty}(\mathcal{U})$ to $\mathcal{C}^{\infty}(\mathcal{U})$ which naturally extends the projection of $\Lambda(V_1^*)$ to \mathbb{R} .

To help the reader, we give two typical examples.

Let $V = \mathbb{R}^{(1,0)}$. Then $V_0 = V = \mathbb{R}$, and $V_{\mathcal{P}} = \mathcal{P}_0$. Let $\mathcal{U} \subset \mathbb{R}$ be an open subset, $\phi \in \mathcal{C}_V^{\infty}(\mathcal{U}, W) = \mathcal{C}^{\infty}(\mathcal{U}, W)$, and $\alpha \in V_{\mathcal{P}} = \mathcal{P}_0$ such that $\mathbf{b}(\alpha) \in \mathcal{U}$. Then $\phi(\alpha) \in \mathcal{P}$ is defined by formula (3).

Let $V = \mathbb{R}^{(0,1)}$. Then $V_0 = \{0\}$, and $V_{\mathcal{P}} = \mathcal{P}_1$. Any $\phi \in \mathcal{C}_V^{\infty}(\{0\}, W)$ can be written as $\phi = c + \xi d$, where c and d are elements of W , and ξ the standard coordinate (the identity function) on \mathbb{R} . Then, for $\alpha \in \mathcal{P}_1$, $\phi(\alpha) \in \mathcal{P}$ is defined by formula

$$\phi(\alpha) = c + \alpha d.$$

Let \mathcal{A} be a commutative superalgebra. We still use the notation $V_{\mathcal{A}} = (V \otimes \mathcal{A})_0$. Polynomial functions $S(V^*)$ can be evaluated on $V_{\mathcal{A}}$, but, in general, smooth functions can be evaluated on $V_{\mathcal{A}}$ only if \mathcal{A} is a near algebra.

The particular case $\mathcal{A} = S(V^*)$ is important, because $V_{\mathcal{A}}$ contains a particular point, the *generic point*, corresponding to the identity in the identification of $\text{Hom}(V, V)_0$ with $(V \otimes V^*)_0 \subset V_{\mathcal{A}}$. Let us call v the generic point. Then we have $f(v) = f$ for any polynomial function $f \in S(V^*)$.

1.4. Coordinates and integration. Let V be a finite dimensional supervector space. By a basis $(g_i)_{i \in I}$ of V , we mean an indexed basis consisting of homogeneous elements. The *dual basis* $(z^i)_{i \in I}$ of V^* is defined by the usual relation $z^j(g_i) = \delta_i^j$ (the Dirac symbol). We will also say that the basis $(g_i)_{i \in I}$ is the *predual basis* of the basis $(z^i)_{i \in I}$ (the dual basis, in the canonical identification of V to the dual of V^* is $((-1)^{p(g_i)} g_i)_{i \in I}$). A basis $(z^i)_{i \in I}$ of V^* will be also called a *system of coordinates on V* . The corresponding vector fields (i.e. derivations of the algebra of smooth functions on V) are denoted by $\frac{\partial}{\partial z^i}$. They are characterized by the rule

$$\frac{\partial}{\partial z^j}(z^i) = \delta_j^i.$$

Notice that the generic point v of V is then given by the formula

$$(5) \quad v = g_i z^i \in V_{S(V^*)}.$$

We will mainly use standard coordinates. Let $(m, n) = \dim(V)$. Then they are basis of V^* of the form $(x^1, \dots, x^m, \xi^1, \dots, \xi^n)$, where (x^1, \dots, x^m) is a basis of V_0^* , and (ξ^1, \dots, ξ^n) a basis of V_1^* . Such a basis will be sometimes denoted by the symbol (x, ξ) . For the corresponding predual basis $(e_1, \dots, e_m, f_1, \dots, f_n)$ of V , then (e_1, \dots, e_m) is a basis of V_0 and (f_1, \dots, f_n) is a basis of V_1 . These notations will be used in particular for the canonical basis of $\mathbb{R}^{(m, n)}$. Let $I = (i_1, \dots, i_n) \in \{0, 1\}^n$. Then we denote by ξ^I the monomial $(\xi^1)^{i_1} \dots (\xi^n)^{i_n}$ of $S(V^*)$. Let $\mathcal{U} \subset V_0$ be an open set. Let W be a Fréchet supervector space. Then any $\phi \in \mathcal{C}_V^\infty(\mathcal{U}, W)$ is of the form

$$(6) \quad \phi = \sum_I \xi^I \phi_I(x^1, \dots, x^m),$$

with ϕ_I is an ordinary W -valued smooth function defined in the appropriate open subset of \mathbb{R}^m . Notice that $\phi_{\mathbb{R}} = \phi_{(0, \dots, 0)}(x^1, \dots, x^m)$ does not depend on the choice of the odd coordinates ξ^i . We emphasize the fact that we write ϕ_I to the right of ξ^I (recall that $\xi^I \phi_I = \pm \phi_I \xi^I$, according to the sign rule).

We will denote by $\mathcal{C}_{V,c}^\infty(\mathcal{U}, W)$ the subspace of $\mathcal{C}_V^\infty(\mathcal{U}, W)$ of function with compact support. Then the *distributions on V defined in \mathcal{U}* are the elements of the (Schwartz's) dual of $\mathcal{C}_{V,c}^\infty(\mathcal{U})$. If t is a distribution, we will use the notation

$$t(\phi) = \int_V t(v) \phi(v)$$

for $\phi \in \mathcal{C}_{V,c}^\infty(\mathcal{U})$. We will also use complex valued distributions, defined in an obvious way.

A *Berezin integral* (or *Haar*, or *Lebesgue*) is by definition a distribution on V which is invariant by translations (i.e. which vanishes on functions of the form $\partial_X \phi$ where $\phi \in \mathcal{C}_{V,c}^\infty(V)$ and ∂_X is the vector field on V with constant coefficients corresponding to $X \in V$: for $f \in V^*$, $\partial_X f = (-1)^{p(X)p(f)} f(X)$).

Let t be such a Berezin distribution. Let $\phi \in \mathcal{C}_{V,c}^\infty(V)$. Let \mathcal{P} be a near superalgebra and $a \in V_{\mathcal{P}}$, then the function $\phi_a(v) = \phi(v+a)$ is a well defined function $\phi_a \in \mathcal{C}_{V,c}^\infty(V, \mathcal{P})$. Then, Taylor formula (3) and invariance by translation of t implies that:

$$(7) \quad \int_V t(v) \phi_a(v) = \int_V t(v) \phi(v+a) = \int_V t(v) \phi(v).$$

Up to a multiplicative constant, there is exactly one Berezin integral, and it is an important matter in this article to choose a particular one for the symplectic oriented supervector spaces (see below).

A choice of a standard system of coordinates determines a specific choice $d_{(x,\xi)}$ of a Berezin integral by the formula

$$(8) \quad \begin{aligned} \int_V d_{(x,\xi)}(v)\phi(v) &= (-1)^{\frac{n(n-1)}{2}} \int_{\mathbb{R}^m} |dx^1 \dots dx^m| \phi_{(1,\dots,1)}(x^1, \dots, x^n) \\ &= \int_{\mathbb{R}^m} |dx^1 \dots dx^m| \left(\frac{\partial}{\partial \xi^1} \dots \frac{\partial}{\partial \xi^n} \phi \right)_{\mathbb{R}}(x^1, \dots, x^m), \end{aligned}$$

for $\phi \in \mathcal{C}_{V,c}^\infty(V_0)$, where $|dx^1 \dots dx^m|$ is the Lebesgue measure on \mathbb{R}^m . Note that this formula can also be applied to any $\phi \in \mathcal{C}_{V,c}^\infty(\mathcal{U}, W)$ with a result in W .

The choice of sign is such that Fubini's formula holds. More precisely, let V, W be two supervector spaces of dimensions (m, n) and (p, q) . Let $(x^1, \dots, x^m, \xi^1, \dots, \xi^n)$ be standard coordinates on V and $(y^1, \dots, y^p, \eta^1, \dots, \eta^q)$ be standard coordinates on W . Then $(x, y, \xi, \eta) = (x^1, \dots, x^m, y^1, \dots, y^p, \xi^1, \dots, \xi^n, \eta^1, \dots, \eta^q)$ defines standard coordinates on $V \times W$. Let $\phi(v, w)$ is a smooth compactly supported function on $V \times W$. Then:

$$(9) \quad \int_{V \times W} d_{(x,y,\xi,\eta)}(v, w)\phi(v, w) = \int_V d_{(x,\xi)}(v) \left(\int_W d_{(y,\eta)}(w)\phi(v, w) \right).$$

We write:

$$(10) \quad d_{(x,y,\xi,\eta)}(v, w) = d_{(x,\xi)}(v) d_{(y,\eta)}(w).$$

In particular, since $V = V_0 \oplus V_1$, $d_{x,\xi} = d_x d_\xi$ and formula (8) is a particular case of formula (9).

Let us stress that if V_1 is not $\{0\}$, in the setting of supermanifolds there is no natural notion of measure on V and no natural notion of positive distribution on V . Thus we use these notions only for (ungraded) vector spaces, or for the even part V_0 of a supervector space, which is then regarded as an ungraded vector space. Otherwise, we use the terms *distribution* or *integral*.

In this article, we will be in fact interested by complex valued distributions. Then we allow standard basis $(e_1, \dots, e_m, f_1, \dots, f_n)$ of $V \otimes \mathbb{C}$, where (e_1, \dots, e_m) is a basis of V_0 and (f_1, \dots, f_n) is a basis of $V_1 \otimes \mathbb{C}$. Then the dual basis (x, ξ) provides a coordinate system (x) on V_0 and a dual basis (ξ) of $V_1^* \otimes \mathbb{C}$. Any $f \in \mathcal{C}_{V,c}^\infty(V_0, \mathbb{C})$ can be written in the form (6), and the (complex) Berezin integral $d_{(x,\xi)}$ is again well defined by formula (8).

1.5. Generalized functions.

1.5.1. *Definition.* Let V be a finite dimensional supervector space and $\mathcal{U} \subset V_0$ be an open set. Let (x, ξ) be a standard coordinates system on V . As usual, we will say that a distribution t on V defined in \mathcal{U} is *smooth* (resp. *smooth compactly supported*) if there is a function $\psi \in \mathcal{C}_V^\infty(\mathcal{U})$ (resp. $\psi \in \mathcal{C}_{V,c}^\infty(\mathcal{U})$) such that $t(v) = d_{x,\xi}(v)\psi(v)$. It means that for any $\phi \in \mathcal{C}_V^\infty(\mathcal{U})$:

$$(11) \quad t(\phi) = \int_V d_{x,\xi}(v)\psi(v)\phi(v).$$

This definition does not depend on the standard coordinates system (x, ξ) .

By definition, the *generalized functions on V defined on \mathcal{U}* are the elements of the (Schwartz's) dual of the space of smooth compactly supported distributions. For a generalized function ϕ and a smooth compactly supported distribution t , we write:

$$(12) \quad \phi(t) = (-1)^{p(t)p(\phi)} \int_V t(v)\phi(v).$$

(the spaces of distributions and thus of generalized functions are naturally $\mathbb{Z}/2\mathbb{Z}$ -graded.)

We denote by $\mathcal{C}^{-\infty}(\mathcal{U})$ the space of generalized functions on \mathcal{U} and by $\mathcal{C}_V^{-\infty}(\mathcal{U})$ the space of generalized functions on V defined on \mathcal{U} .

Let us remark that, as $\mathcal{C}_V^\infty(\mathcal{U}) = \mathcal{C}^\infty(\mathcal{U}) \otimes \Lambda(V_1^*)$, we have:

$$(13) \quad \mathcal{C}_V^{-\infty}(\mathcal{U}) = \mathcal{C}^{-\infty}(\mathcal{U}) \otimes \Lambda(V_1^*).$$

Let W be a Fréchet supervector space. A W -valued generalized function is a continuous homomorphism (in sense of Schwartz) from the space of smooth compactly supported distributions to W . We denote by $\mathcal{C}_V^{-\infty}(\mathcal{U}, W)$ the set of W -valued generalized functions. If W is finite dimensional, we have $\mathcal{C}_V^{-\infty}(\mathcal{U}, W) = \mathcal{C}_V^{-\infty}(\mathcal{U}) \otimes W$. We will be in particular concerned with the cases $W = \mathbb{C}$ and $W = \Lambda(E^*)$ for some finite dimensional vector space E .

1.5.2. Wave front set.

Definition 1.1. Let V be a supervector space. Let $\mathcal{U} \subset V_0$ be an open subset of V_0 . Let $\psi \in \mathcal{C}_V^{-\infty}(\mathcal{U})$ be a generalized function on V defined on \mathcal{U} . Let (ξ^1, \dots, ξ^n) be a basis of V_1^* . We put $\psi = \sum_I \xi^I \psi_I$ where ψ_I is a generalized function on \mathcal{U} .

The wave front set $WF(\psi)$ of ψ , is by definition the union of the wave front sets of the ψ_I :

$$(14) \quad WF(\psi) = \bigcup_I WF(\psi_I) \subset T^*\mathcal{U},$$

where $WF(\psi_I)$ is the wave front set of ψ_I and $T^*\mathcal{U} = \mathcal{U} \times V_0^*$ is the cotangent bundle of \mathcal{U} .

The definition of $WF(\psi)$ does not depends on the choice the basis (ξ^i) of V_1^* .

1.6. Rapidly decreasing functions. We say (cf. for example [Hör83, Chapter 7]) that $\phi \in \mathcal{C}^\infty(\mathbb{R}^m)$ is rapidly decreasing if for any $(\alpha_1, \dots, \alpha_m) \in \mathbb{N}^m$ and any $(\beta_1, \dots, \beta_m) \in \mathbb{N}^m$,

$$(15) \quad \text{Sup} \left| (x^1)^{\beta_1} \dots (x^m)^{\beta_m} \frac{\partial^{\alpha_1}}{\partial (x^1)^{\alpha_1}} \dots \frac{\partial^{\alpha_m}}{\partial (x^m)^{\alpha_m}} \phi \right| < +\infty.$$

where (x^1, \dots, x^m) are the canonical coordinates on \mathbb{R}^m .

Definition 1.2. Let V be a supervector space. Let $\phi \in \mathcal{C}_V^\infty(V_0)$ be a smooth function on V . Let $(x^1, \dots, x^m, \xi^1, \dots, \xi^n)$ be a basis of V^* . We put $\phi = \sum_I \xi^I \phi_I(x^1, \dots, x^m)$ where $\phi_I \in \mathcal{C}^\infty(\mathbb{R}^m)$.

We say that ϕ is rapidly decreasing if for any I , ϕ_I is a rapidly decreasing function on \mathbb{R}^m .

This definition does not depend on the choice of the basis (x^i, ξ^j) of V^* .

1.7. Holomorphic functions. Let V be a complex finite dimensional supervector space. Let \mathcal{U} be an open subset of V_0 . Let W be a complex Fréchet supervector space. By definition, an holomorphic function on V with values in W defined on \mathcal{U} is an holomorphic function on \mathcal{U} with values in $\Lambda(V_1^*) \otimes W$. We denote by $\mathcal{H}_V(\mathcal{U}, W)$ (resp. $\mathcal{H}(V, W)$) the algebra of holomorphic functions on V (resp. on V_0) with values in W defined on \mathcal{U} :

$$(16) \quad \mathcal{H}_V(\mathcal{U}, W) = \mathcal{H}(\mathcal{U}, W) \otimes \Lambda(V_1^*).$$

Let \mathcal{P} be a (real) near superalgebra. We denote by $\mathcal{P}_{\mathbb{C}} = \mathcal{P} \otimes \mathbb{C}$ its complexification. We put $V_{\mathcal{P}} = (V \otimes \mathcal{P}_{\mathbb{C}})_0$ and $W_{\mathcal{P}} = (W \otimes \mathcal{P}_{\mathbb{C}})_0$. As in the real case we have a body map $\mathbf{b} : V_{\mathcal{P}} \rightarrow V_0$ which extends the canonical projection $\mathbf{b} \otimes 1 : \mathcal{P} \otimes \mathbb{C} \rightarrow \mathbb{R} \otimes \mathbb{C} \simeq \mathbb{C}$. We put for \mathcal{U} open in V_0 : $V_{\mathcal{P}}(\mathcal{U}) = \{v \in V_{\mathcal{P}} / \mathbf{b}(v) \in \mathcal{U}\}$.

Let $\phi \in \mathcal{H}_V(\mathcal{U}, W)$ and $\alpha \in V_{\mathcal{P}}(\mathcal{U})$ as in the real case, we denote by $\phi(\alpha) \in W_{\mathcal{P}}$ the image of α by ϕ defined by a formula analogous to formula (3). The map $\alpha \mapsto \phi(\alpha)$ defines an holomorphic function $\phi_{\mathcal{P}} \in \mathcal{H}(V_{\mathcal{P}}, W_{\mathcal{P}})$ on $V_{\mathcal{P}}$ with values in $W \otimes \mathcal{P}_{\mathbb{C}}$.

As for smooth functions on a real supervector space, if \mathcal{P} is large enough, the map $\phi \mapsto \phi_{\mathcal{P}}$ is injective.

1.8. Analytic functions. Let V be a real finite dimensional supervector space. Let \mathcal{U} be an open subset of V_0 . Let W be a real supervector space. We denote by $\mathcal{C}^{\omega}(\mathcal{U}, W)$ the set of analytic functions on \mathcal{U} with values in W . We put:

$$\mathcal{C}_V^{\omega}(\mathcal{U}, W) = \mathcal{C}^{\omega}(\mathcal{U}, W) \otimes \Lambda(V_1^*).$$

We call the elements of $\mathcal{C}_V^{\omega}(\mathcal{U}, W) \subset \mathcal{C}_V^{\infty}(\mathcal{U}, W)$ the analytic functions on V with values in W defined on \mathcal{U} .

As usual we put $\mathcal{C}^{\omega}(V, W) = \mathcal{C}_V^{\omega}(V_0, W)$ and $\mathcal{C}_V^{\omega}(\mathcal{U}) = \mathcal{C}_V^{\omega}(\mathcal{U}, \mathbb{R})$ (resp. $\mathcal{C}^{\omega}(V) = \mathcal{C}^{\omega}(V, \mathbb{R})$).

1.9. Supertrace and Berezinians. Let V be a supervector space. We denote by $\mathfrak{gl}(V)$ the Lie superalgebra of endomorphisms of V .

Let \mathcal{A} be a commutative superalgebra. We write an element of $\mathfrak{gl}(V)_{\mathcal{A}}$ in the form

$$(17) \quad M = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \mathfrak{gl}(V)_{\mathcal{A}}.$$

where $A \in \mathfrak{gl}(V_0) \otimes \mathcal{A}_0$, $D \in \mathfrak{gl}(V_1) \otimes \mathcal{A}_0$, $B \in \text{Hom}(V_1, V_0) \otimes \mathcal{A}_1$, and $C \in \text{Hom}(V_0, V_1) \otimes \mathcal{A}_1$.

We recall the definition of the supertrace:

Definition 1.3. *The supertrace of $M \in \mathfrak{gl}(V)_{\mathcal{A}}$ is defined by*

$$(18) \quad \text{str}(M) = \text{tr}(A) - \text{tr}(D),$$

where tr is the ordinary trace.

Berezin introduced the following generalizations of the determinant (cf. [Ber87, BL75, Man88]), called the *Berezinian* and *inverse Berezinian*.

Definition 1.4. *If D is invertible, we define:*

$$(19) \quad \text{Ber}(M) = \det(A - BD^{-1}C) \det(D)^{-1},$$

and if A is invertible:

$$(20) \quad \text{Ber}^{-}(M) = \det(A)^{-1} \det(D - CA^{-1}B).$$

Definition 1.5. Assume moreover that \mathcal{A} is a near superalgebra. If D is invertible, we define (cf. [Vor91]):

$$(21) \quad \text{Ber}_{(1,0)}(M) = \left| \det(A - BD^{-1}C) \right| \det(D)^{-1},$$

and if A is invertible:

$$(22) \quad \text{Ber}_{(1,0)}^{-}(M) = \left| \det(A)^{-1} \right| \det(D - CA^{-1}B),$$

All these functions are multiplicative, and when both A and D are invertible, it is known that $\text{Ber}^{-}(M) = \text{Ber}^{-1}(M)$ and $\text{Ber}_{(1,0)}^{-}(M) = \text{Ber}_{(1,0)}^{-1}(M)$.

Recall that $\mathfrak{gl}(V)_0$ consists of the matrices $\begin{pmatrix} A & 0 \\ 0 & D \end{pmatrix}$ with $A \in \mathfrak{gl}(V_0)$ and $D \in \mathfrak{gl}(V_1)$.

We consider the two open sets $\mathcal{U}' = GL(V_0) \times \mathfrak{gl}(V_1)$, and $\mathcal{U}'' = \mathfrak{gl}(V_0) \times GL(V_1)$. Formula (19) defines a rational function on the open set \mathcal{U}'' of the supermanifold $\mathfrak{gl}(V)$. Formula (21) defines a smooth function on the open set \mathcal{U}'' of the supermanifold $\mathfrak{gl}(V)$. We still denote by Ber and $\text{Ber}_{(1,0)}$ the elements of $\mathcal{C}_{\mathfrak{gl}(V)}^{\infty}(\mathcal{U}'')$ whose evaluation in $\mathfrak{gl}(V)_{\mathcal{A}}$ is given as above. We similarly define the elements Ber^{-} and $\text{Ber}_{(1,0)}^{-}$ of $\mathcal{C}_{\mathfrak{gl}(V)}^{\infty}(\mathcal{U}')$.

1.10. Symplectic oriented supervector spaces. Let $V = V_0 \oplus V_1$ be a supervector space. A *symplectic form* B on V is a non degenerate even skew symmetric bilinear form on V . It means that V_0 and V_1 are orthogonal, that the restriction of B to V_0 is a non degenerate skew symmetric bilinear form, and that the restriction of B to V_1 is a non degenerate symmetric bilinear form. We call V , provided with B , a *symplectic supervector space*.

Such a space is a direct sum of $(2, 0)$ -dimensional symplectic supervector spaces (i.e. 2-dimensional symplectic vector spaces), and of $(0, 1)$ -dimensional symplectic supervector spaces (i.e. 1-dimensional quadratic vector spaces). We first review these building blocks.

1.10.1. Symplectic 2-dimensional vector spaces. Let $V = V_0$ a purely even 2-dimensional symplectic space. A *symplectic basis* of V is a basis (e_1, e_2) such that $B(e_1, e_2) = 1$, $B(e_1, e_1) = 0$, $B(e_2, e_2) = 0$. It defines a dual *symplectic coordinate system* (x^1, x^2) , an orientation of V^* , and a *Liouville integral* (a particular normalization of the Berezin integral):

$$\phi \in \mathcal{C}_c^{\infty}(V) = \mathcal{C}_c^{\infty}(\mathbb{R}^2) \mapsto \int_V d_V(v) \phi(v) = \frac{1}{2\pi} \int |dx^1 dx^2| \phi(x^1, x^2).$$

1.10.2. Symplectic 1-dimensional odd vector spaces. Let $V = V_1$ a purely odd 1-dimensional symplectic supervector space (i.e. a 1-dimensional quadratic space). A symplectic basis of V is a basis (f) such that $B(f, f) = 1$. However, such a basis does not always exists, and we allow f to be in $V_1 \cup iV_1 \subset V_1 \otimes \mathbb{C}$. Let $(\xi) \in V_1^* \cup iV_1^*$ be the dual basis. It defines a *Liouville integral* (which is complex valued if B is negative definite) d_V :

$$\phi = a + \xi b \in \Lambda(V^* \otimes \mathbb{C}) \mapsto \int_V d_V(v) \phi(v) = b.$$

We will call the choice of (ξ) (the other possible choice is $(-\xi)$) an *orientation* of V_1 . If B is positive definite, then (ξ) is a basis of V_1^* , and so defines an orientation in the usual sense. If B is negative definite, then $(-i\xi)$ is a basis of V_1 , and so defines an orientation in the usual sense.

1.10.3. *General case.* Let us go back to the general case. Since V_0 is a classical symplectic space, there is a canonical normalization of Lebesgue integral on V_0 , the Liouville integral, which we recall now.

The dimension m of V_0 is even. We choose a symplectic basis (e_1, \dots, e_m) of V_0 , that is V_0 is the direct sum of $m/2$ symplectic vector spaces generated by the pairs $(e_1, e_2), (e_3, e_4), \dots$, and $B(e_1, e_2) = 1, B(e_3, e_4) = 1, \dots$. The dual basis (x^i) of V_0^* is called a symplectic coordinate system. The Liouville integral on V_0 is

$$\frac{1}{(2\pi)^{m/2}} |dx^1 \dots dx^m|.$$

The Liouville integral does not depend on the choice of the symplectic basis of V_0 .

On V_1 , we define a symplectic basis (f_1, \dots, f_n) as an orthonormal basis of $V_1 \otimes \mathbb{C}$ such that $f_i \in V_1$ or $f_i \in iV_1$ for all i . Let (ξ^1, \dots, ξ^n) be the dual basis. The pair of functions $\pm \xi^1 \dots \xi^n$ does not depend on the symplectic basis (f_1, \dots, f_n) . A choice of one of the two elements of $\pm \xi^1 \dots \xi^n$ is called an *orientation* of V_1 . If V_1 is oriented, an oriented symplectic coordinate system on V_1 is a basis for which the orientation is $\xi^1 \dots \xi^n$.

We call the corresponding Berezin integral d_ξ the Liouville integral of the oriented symplectic space V_1 .

Let us remark that in the specially interesting case where B is positive definite, a symplectic basis is a basis of V_1 , and not only of $V_1 \otimes \mathbb{C}$.

We define an *oriented symplectic supervector space* as a symplectic supervector space (V, B) provided with an orientation of V_1 . A symplectic oriented basis $(e_1, \dots, e_m, f_1, \dots, f_n)$ is a basis of $V \otimes \mathbb{C}$ such that (e_1, \dots, e_m) is a symplectic basis of V_0 , and (f_1, \dots, f_n) an oriented symplectic basis of $V_1 \otimes \mathbb{C}$. We use the corresponding dual system of coordinates (x, ξ) , and the associated Berezin integral $d_{(x, \xi)}$ will be denoted by d_V . Then V becomes a symbol bearing a supermanifold structure, a symplectic structure, an orientation...

1.11. **Symplectic Lie superalgebras.** Let $V = (V, B)$ be a symplectic supervector space. We denote by $\mathfrak{spo}(V, B)$ (or $\mathfrak{spo}(V)$) the Lie subsuperalgebra of $\mathfrak{gl}(V)$ consisting of endomorphisms of V which leave B invariant. We call it the *symplectic Lie superalgebra*—it is usually called *orthosymplectic*—.

We have $\mathfrak{spo}(V, B)_0 = \mathfrak{sp}(V_0) \oplus \mathfrak{so}(V_1)$.

A particular linear even isomorphism μ of $\mathfrak{spo}(V)$ to $S^2(V^*)$, called the moment mapping, will play an important role. Thus μ is an element of $S(\mathfrak{spo}(V)^* \otimes V^*)$, and we consider it as a function on the supermanifold $\mathfrak{spo}(V) \times V$, linear in the first variable, polynomial of degree 2 in the second variable, and globally homogeneous of degree 3. It is defined by the formula

$$\mu(X, v) = -\frac{1}{2} B(v, Xv),$$

where, for any commutative superalgebra \mathcal{A} , X and v are \mathcal{A} -valued points of $\mathfrak{spo}(V)$ and V , and $B(v, Xv) \in \mathcal{A}$ is defined by the natural extension of scalars. Considering a basis G_k of $\mathfrak{spo}(V)$, the dual basis Z^k , the generic point $X = G_k Z^k$, a basis g_i of V , the dual basis z^i , and the generic point $v = g_i z^i$, we obtain:

$$\mu = -\frac{1}{2} B(g_i, G_k g_j) z^j Z^k z^i.$$

We will also consider e^μ , which is a smooth (and even analytic) function on the supermanifold $\mathfrak{spo}(V) \times V$.

Let us explain the choice of the constant $-\frac{1}{2}$ in definition of μ and why we call μ the moment mapping.

The symplectic form on V gives to the associated supermanifold a structure of symplectic supermanifold. We define a Poisson bracket on $S(V^*)$ by the following. Let $f \in V^*$, we denote by v_f the element of V such that for any $w \in V$:

$$(23) \quad B(v_f, w) = f(w).$$

This gives an isomorphism from V^* onto V . For $f, g \in V^*$, we put:

$$(24) \quad \{f, g\} = B(v_f, v_g)$$

and we extend it to a Poisson bracket on $S(V^*)$.

Let $\check{\mu}$ be the linear form on \mathfrak{g} with values in $S^2(V^*)$ such that $\check{\mu}(X)(v) = \mu(X, v)$.

With the above definitions we have:

$$(25) \quad \{\check{\mu}(X), \check{\mu}(Y)\} = \check{\mu}([X, Y]).$$

Thus $\check{\mu}$ is a morphism of Lie algebras.

We extend $\check{\mu}$ to a morphism of superalgebras from $S(\mathfrak{spo}(V))$ to $S(V^*)$. More precisely, we put for $X_1 \dots X_k \in S^k(\mathfrak{spo}(V))$:

$$(26) \quad \check{\mu}(X_1 \dots X_k) = \check{\mu}(X_1) \dots \check{\mu}(X_k).$$

Its image is $\bigoplus_{k \in \mathbb{N}} S^{2k}(V^*)$. Indeed, the natural morphism $S^k(S^2(V^*)) \rightarrow S^{2k}(V^*)$ is surjective. Moreover, $\check{\mu} : \mathfrak{spo}(V) \rightarrow S^2(V^*)$ is bijective and by definition $\check{\mu}$ factorises:

$$(27) \quad S^k(\mathfrak{spo}(V)) \rightarrow S^k(S^2(V^*)) \rightarrow S^{2k}(V^*).$$

We put $\mathcal{I}_k = \ker(\check{\mu}) \cap S^k(\mathfrak{spo}(V))$. We have: $\ker(\check{\mu}) = \bigoplus_{k \in \mathbb{N}} \mathcal{I}_k$. We choose a supplementary $\mathcal{S}_k \subset S^k(\mathfrak{spo}(V))$ of \mathcal{I}_k :

$$(28) \quad S^k(\mathfrak{spo}(V)) = \mathcal{I}_k \oplus \mathcal{S}_k.$$

Let

$$(29) \quad \mathcal{S} = \bigoplus_{k \in \mathbb{N}} \mathcal{S}_k,$$

we have $S(\mathfrak{spo}(V)) = \mathcal{S} \oplus \ker(\check{\mu})$.

Then the restriction $\check{\mu} : \mathcal{S} \rightarrow \bigoplus_{k \in \mathbb{N}} S^{2k}(V^*)$ is bijective. We denote by

$$\Xi : \bigoplus_{k \in \mathbb{N}} S^{2k}(V^*) \rightarrow \mathcal{S} \subset S(\mathfrak{spo}(V))$$

its inverse. For $P \in S^{2k}(V^*)$, $\Xi(P) \in \mathcal{S}_k \subset S^k(\mathfrak{spo}(V))$ and

$$(30) \quad \check{\mu}(\Xi(P)) = P.$$

2. SUPERPFAFFIAN I : AN ANALYTIC FUNCTION

Let $V = V_0 \oplus V_1$ be an oriented supervector symplectic space of dimension (m, n) . We already defined the symplectic integral d_V and the moment map μ . We now define some open subsets of $\mathfrak{spo}(V)_0$ in which we want to define functions on $\mathfrak{spo}(V)$.

For $X \in \mathfrak{sp}(V_0)$, $v \mapsto B(v, Xv)$ is a quadratic form on V_0 . We denote by $\mathcal{U} \subset \mathfrak{sp}(V_0)$ the open set of $X \in \mathfrak{sp}(V_0)$ for which this form is non degenerate (or equivalently, $X \in \mathfrak{sp}(V_0) \subset \mathfrak{gl}(V_0)$ is invertible). It is the disjoint union of the subsets $\mathcal{U}_{p,q}$ (with $p + q = m$) where (p, q) is the signature of the quadratic form. Let us also use the notation $\mathcal{U}^+ = \mathcal{U}_{m,0}$ (resp. $\mathcal{U}^- = \mathcal{U}_{0,m}$) for the open set of X for which it is positive definite (resp. negative definite). It is an open convex cone in $\mathfrak{sp}(V_0)$. We denote by $\mathcal{V} = \mathcal{U} \times \mathfrak{so}(V_1)$, $\mathcal{V}_{p,q} = \mathcal{U}_{p,q} \times \mathfrak{so}(V_1)$, $\mathcal{V}^+ = \mathcal{U}^+ \times \mathfrak{so}(V_1)$, $\mathcal{V}^- = \mathcal{U}^- \times \mathfrak{so}(V_1)$ the corresponding open subsets of $\mathfrak{spo}(V)_0$. Then \mathcal{V}^+ is an open convex cone in $\mathfrak{spo}(V)_0$.

In this section, we prove:

Theorem 2.1. *There exists a unique function*

$$\text{Spf} \in \mathcal{C}_{\mathfrak{spo}(V)}^\omega(\mathcal{V}^+, \mathbb{C})$$

such that for any near superalgebra \mathcal{P} and any element X of $\mathfrak{spo}(V)_{\mathcal{P}}$ such that the body $\mathbf{b}(X)$ is in \mathcal{V}^+ , (that is, with the notations of section 1.3, $X \in \mathfrak{spo}(V)_{\mathcal{P}}(\mathcal{V}^+)$):

$$(31) \quad \text{Spf}(X) = \int_V d_V(v) \exp(\mu(X, v)) = \int_V d_V(v) \exp(-\frac{1}{2}B(v, Xv)).$$

We will prove it theorem in section 2.1. We call the function Spf defined in the preceding theorem the *superPfaffian*. We begin by giving typical examples.

2.0.1. Symplectic 2-dimensional vector spaces. Let $V = V_0$ a purely even 2-dimensional symplectic space (cf. subsection 1.10.1). Choose a symplectic basis (e_1, e_2) of V . The dual coordinate system will be denoted by (x, y) (instead of (x^1, x^2)). An element $X \in \mathfrak{spo}(V)$ is represented by a matrix $X = \begin{pmatrix} a & b \\ c & -a \end{pmatrix} \in \mathfrak{sl}(2, \mathbb{R})$. The corresponding quadratic form $B(v, Xv)$ is equal to $B(e_1x + e_2y, X(e_1x + e_2y)) = B(e_1x + e_2y, (e_1a + e_2c)x + (e_1b - e_2a)y) = x(cx - ay) - y(ax + by) = cx^2 - 2axy - by^2$. Thus, if $v = e_1x + e_2y$ is the generic point of V :

$$(32) \quad \mu(X, v) = \frac{-1}{2}(cx^2 - 2axy - by^2).$$

The set \mathcal{U}^+ is defined by the equations $\det(X) = -a^2 - bc > 0$, and $c > 0$. It is one of the two connected components of the set of invertible elliptic matrices. For example, the matrix $\begin{pmatrix} 0 & -c \\ c & 0 \end{pmatrix}$ is in \mathcal{U}^+ if $c > 0$, and these matrices are representatives for the conjugacy classes of $SL(2, \mathbb{R})$ in \mathcal{U}^+ .

For $X \in \mathcal{U}^+$ the integral

$$(33) \quad \text{Spf}(X) = \int_V d_V(v) \exp(-\frac{1}{2}B(v, Xv))$$

is convergent and defines an analytic function on \mathcal{U}^+ . By a suitable change of variable we see that the function Spf is invariant by $SL(2, \mathbb{R})$. For $X = \begin{pmatrix} 0 & -c \\ c & 0 \end{pmatrix}$ with $c > 0$, we

get

$$\text{Spf}(X) = \frac{1}{2\pi} \int |dx dy| \exp(-\frac{1}{2}c(x^2 + y^2)),$$

and so

$$(34) \quad \text{Spf}\left(\begin{pmatrix} 0 & -c \\ c & 0 \end{pmatrix}\right) = 1/c \quad \text{if } c > 0.$$

Notice also that $\text{Ber}^-(X) = 1/\det(X) = 1/c^2$.

Our conclusion is that *on \mathcal{U}^+ the inverse Berezinian is positive, and that Spf is the positive square root of the inverse Berezinian.*

Consider now a near superalgebra \mathcal{P} . Let $X \in \mathfrak{sl}(2, \mathbb{R}) \otimes \mathcal{P}_0$ such that $\mathbf{b}(X) \in \mathcal{U}^+$. We write $X = \begin{pmatrix} a + \alpha & b + \beta \\ c + \gamma & -a - \alpha \end{pmatrix}$, with a, b, c real such that $\det(X) = -a^2 - bc > 0$ and $c > 0$, and with α, β, γ nilpotent elements of \mathcal{P}_0 . Then, for $v = e_1x + e_2y$, we obtain

$$(35) \quad \begin{aligned} \exp(-\frac{1}{2}B(v, Xv)) &= \exp(-\frac{1}{2}((c + \gamma)x^2 - 2(a + \alpha)xy - (b + \beta)y^2)) \\ &= \exp(-\frac{1}{2}(cx^2 - 2axy - by^2)) \exp(-\frac{1}{2}(\gamma x^2 - 2\alpha xy - \beta y^2)). \end{aligned}$$

Since α, β and γ are nilpotent the last exponential is a polynomial on V with values in \mathcal{P}_0 . It follows that $\int_V d_V(v) \exp(-\frac{1}{2}B(v, Xv))$ converges because $cx^2 - 2axy - by^2$ is a positive definite quadratic form on V .

Now, on any compact set included in \mathcal{U}^+ , $\exp(-\frac{1}{2}(cx^2 - 2axy - by^2))$ and its derivatives can be uniformly bounded by a rapidly decreasing function on V . Thus, Spf is a smooth function on \mathcal{U}^+ and its derivatives are obtained by derivation under the summation symbol. It follows that the value of $\text{Spf}\left(\begin{pmatrix} a + \alpha & b + \beta \\ c + \gamma & -a - \alpha \end{pmatrix}\right)$ obtained by applying formula (3) coincides with the value obtained by integration of formula (31). In particular it implies that equality $\text{Spf}(X)^2 = \text{Ber}(X)$ is still valid on $\mathfrak{sp}(V)_{\mathcal{P}}(\mathcal{U}^+)$ (that is for $X = \begin{pmatrix} a + \alpha & b + \beta \\ c + \gamma & -a - \alpha \end{pmatrix}$ with a, b, c real such that $\det(X) = -a^2 - bc > 0$ and $c > 0$, and with α, β, γ nilpotent elements of \mathcal{P}_0).

2.0.2. Symplectic oriented 2-dimensional odd vector spaces. If V is of dimension $(0, 1)$, then $\mathfrak{spo}(V)$ is $\{0\}$, so the next interesting example is when V is of dimension $(0, 2)$.

Then $V = V_1$, and we choose an oriented symplectic basis (f_1, f_2) of $V_1 \otimes \mathbb{C}$. Recall that it means in particular that $B(f_1, f_2) = 0, B(f_1, f_1) = B(f_2, f_2) = 1$. We denote the dual basis by (ξ, η) . Then the orientation is the choice of $\xi\eta \in S^2(V^*) = \Lambda^2(V_1^*)$ (versus $-\xi\eta$). In this case the superPfaffian is the ordinary Pfaffian, a polynomial (and in fact, a linear) function on $\mathfrak{so}(V)$ and so it is also defined on $\mathfrak{so}(V \otimes \mathbb{C})$. An element X of $\mathfrak{so}(V \otimes \mathbb{C})$ is represented in the given basis by a matrix $X = \begin{pmatrix} 0 & -c \\ c & 0 \end{pmatrix}$ with $c \in \mathbb{C}$. Then, for the generic point $v = f_1\xi + f_2\eta$, we have $B(v, Xv) = B(f_1\xi + f_2\eta, f_2c\xi - f_1c\eta) = 2c\xi\eta$ and $\exp(-\frac{1}{2}B(v, Xv)) = 1 - c\xi\eta$.

We obtain that the integral $\int_V d_{(\xi, \eta)}(v) \exp(-\frac{1}{2}B(v, Xv))$, which by definition is the constant term of the function $\frac{\partial}{\partial \xi} \frac{\partial}{\partial \eta}(1 - c\xi\eta)$, is equal to c , and so

$$(36) \quad \text{Spf}\left(\begin{pmatrix} 0 & -c \\ c & 0 \end{pmatrix}\right) = c \quad \text{if } c \in \mathbb{C}.$$

Notice also that $\text{Ber}^-(X) = \det(X) = c^2$. So again $\text{Spf}(X)$ is a square root of the inverse Berezinian.

If B is positive (resp. negative) definite, $f_1, f_2 \in V_1$ (resp. $\in iV_1$) and thus if $X \in \mathfrak{so}(V_1)$, its matrix in the basis (f_1, f_2) is real and $\text{Spf}(X) \in \mathbb{R}$.

If B is hyperbolic, we can take $f_1 \in V_1$ and $f_2 \in iV_1$ and thus if $X \in \mathfrak{so}(V_1)$, its matrix in the basis (f_1, f_2) is purely imaginary and $\text{Spf}(X) \in i\mathbb{R}$.

2.1. Proof of theorem 2.1. Let \mathcal{P} be any near superalgebra. Then $X \in \mathfrak{spo}(V)_{\mathcal{P}}(\mathcal{V}^+)$ means $X \in \mathfrak{spo}(V)_{\mathcal{P}}$ with $\mathbf{b}(X) \in \mathcal{V}^+$.

We denote by $\text{Spf}_{\mathcal{P}}$ the function on $\mathfrak{spo}(V)_{\mathcal{P}}(\mathcal{V}^+)$ such that for any $X \in \mathfrak{spo}(V)_{\mathcal{P}}(\mathcal{V}^+)$,

$$\text{Spf}_{\mathcal{P}}(X) = \int_V d_V(v) \exp(\mu(X, v)) = \int_V d_V(v) \exp(-\frac{1}{2}B(v, Xv)).$$

2.1.1. $\text{Spf}_{\mathcal{P}}$ is a well defined \mathcal{P} -valued analytic function on $\mathfrak{spo}(V)_{\mathcal{P}}(\mathcal{V}^+)$. Let $X \in \mathfrak{spo}(V)_{\mathcal{P}}(\mathcal{V}^+)$. Let $X_0 \in \mathcal{U}^+$ and $X_1 \in \mathfrak{sp}(V_1)$ such that $\mathbf{b}(X) = X_0 + X_1$. Thus $X = X_0 + X_1 + N$ with N nilpotent.

Let $(e_1, \dots, e_m, f_1, \dots, f_n)$ be a standard basis of V . Let (x, ξ) be the dual basis. Let $v = e_i x^i + f_j \xi^j$ be the generic point of V , $v_0 = e_i x^i$ be the generic point of V_0 and $v_1 = f_j \xi^j$ the generic point of V_1 . We have $v = v_0 + v_1$, $B(v, X_0 v) = B(v_0, X_0 v_0)$ and $B(v, X_1 v) = B(v_1, X_1 v_1)$.

In particular $B(v, X_1 v)$ is nilpotent. Thus $B(v, (X - X_0)v) \in (S^2(V^*) \otimes \mathcal{P})_0$ is nilpotent. It follows that $\exp(-\frac{1}{2}B(v, (X - X_0)v)) \in (S(V^*) \otimes \mathcal{P})_0$ is a polynomial on V with values in \mathcal{P} and that $X \mapsto \exp(-\frac{1}{2}B(v, (X - X_0)v)) \in (S(V^*) \otimes \mathcal{P})_0$ is polynomial on $\mathfrak{spo}(V)_{\mathcal{P}}$.

Let $Z \in \mathfrak{spo}(V)_{\mathcal{P}}$ such that $\mathbf{b}(Z) \in \mathfrak{so}(V_1)$. Then $B(v, Zv)$ is a nilpotent element of $S^2(V^*)_{\mathcal{P}}$. We put:

$$(37) \quad P(Z, v_0) = \int_{V_1} d_{V_1}(v_1) \exp(-\frac{1}{2}B(v_0 + v_1, Z(v_0 + v_1))) \in S(V_0^*) \otimes \mathcal{P}.$$

Now Fubini's formula gives:

$$(38) \quad \int_V d_V(v) \exp(-\frac{1}{2}B(v, Xv)) = \int_{V_0} d_{V_0}(v_0) \exp(-\frac{1}{2}B(v_0, X_0 v_0)) P(X - X_0, v_0).$$

Since $X_0 \in \mathcal{U}^+$, $B(v_0, X_0 v_0)$ is a positive definite quadratic form. The integral on the right hand side is a Gaussian integral on V_0 . Thus $\text{Spf}_{\mathcal{P}}$ is an analytic function on $\mathfrak{spo}(V)_{\mathcal{P}}(\mathcal{V}^+)$.

2.1.2. Spf is a well defined analytic function on \mathcal{V}^+ . Let $\mathcal{Q} = \Lambda(\mathfrak{spo}(V)_1^*)$. Let $h : \mathfrak{spo}(V)_0 \hookrightarrow (\mathfrak{spo}(V)_0)_{\mathcal{Q}}$ be the canonical embedding defined by $h(v) = v \otimes 1$. Let $\Xi \in (\mathfrak{spo}(V)_1)_{\mathcal{Q}}$ be the generic point of $\mathfrak{spo}(V)_1$. We put for $X \in \mathcal{V}^+$:

$$\begin{aligned}
(39) \quad \phi(X) &= \text{Spf}_{\mathcal{Q}}(h(X) + \Xi) \in \mathcal{Q} = \Lambda(\mathfrak{spo}(V)_1^*) \\
&= \int_{V_0} d_{V_0}(v_0) \exp\left(-\frac{1}{2}B(v_0, X_0 v_0)\right) P(X_1 + \Xi, v_0).
\end{aligned}$$

($X = X_0 + X_1$ with $X_0 \in \mathcal{U}^+$ and $X_1 \in \mathfrak{so}(V_1)$; P is defined by (37).) It defines a function

$$(40) \quad \phi \in \mathcal{C}_{\mathfrak{spo}(V)}^{\omega}(\mathcal{V}^+),$$

such that for any near superalgebra \mathcal{P} , any $X = Y + Z \in \mathfrak{spo}(V)_{\mathcal{P}}(\mathcal{V}^+)$ with $Y \in (\mathfrak{spo}(V)_0)_{\mathcal{P}}(\mathcal{V}^+)$ and $Z \in (\mathfrak{spo}(V)_1)_{\mathcal{P}}$, $\phi(X) = \phi(Y)(Z)$.

Since ϕ is defined by a Gaussian integral on V_0 all its derivatives along $\mathfrak{spo}(V)_0$ are determined by derivation “under the integral”. Moreover the above integral is $\Lambda(\mathfrak{spo}(V)_1^*)$ -linear, hence for any near superalgebra \mathcal{P} and $X \in \mathfrak{spo}(V)_{\mathcal{P}}$, $\phi(X) = \text{Spf}_{\mathcal{P}}(X)$.

Now we put $\text{Spf} = \phi \in \mathcal{C}_{\mathfrak{spo}(V)}^{\omega}(\mathcal{V}^+)$ and call it the *superPfaffian*. The preceding remark shows that for any near superalgebra \mathcal{P} and $X \in \mathfrak{spo}(V)_{\mathcal{P}}(\mathcal{V}^+)$ the expression $\text{Spf}(X)$ is not ambiguous: the value of the function Spf at X is given by formula (31).

We will see below (cf. section 2.6.2) that $\text{Spf}^2 = \text{Ber}^-$.

2.2. Holomorphic extension in the appropriate subset. Formula (31) is meaningful for $X \in \mathfrak{spo}(V \otimes \mathbb{C})_{\mathcal{P}}$ with $\mathbf{b}(X) \in \mathcal{V}^+ \times \mathfrak{i spo}(V)_0$ and it defines an holomorphic function on $\mathcal{V}^+ \times \mathfrak{i spo}(V)$.

Indeed, let $X \in \mathfrak{spo}(V \otimes \mathbb{C})_{\mathcal{P}}$ with $\mathbf{b}(X) \in \mathcal{V}^+ \times \mathfrak{i spo}(V)_0$. As before, let $X_0 \in \mathcal{U}^+ \times \mathfrak{i sp}(V_0)$ and $X_1 \in \mathfrak{so}(V_1 \otimes \mathbb{C})$ such that $\mathbf{b}(X) = X_0 + X_1$. The calculations of section 2.1.1 can be reproduced here. The right hand side of formula (38) is still a Gaussian integral and therefore defines a complex analytic function on $\mathfrak{spo}(V \otimes \mathbb{C})_{\mathcal{P}}(\mathcal{V}^+ \times \mathfrak{i spo}(V)_0)$.

The same arguments as in section 2.1.2 show that formula (31) defines an holomorphic extension of Spf on $\mathcal{V}^+ \times \mathfrak{i spo}(V)_0$ still denoted by Spf .

2.3. Invariance. Let \mathcal{P} be a near superalgebra. Let $X \in \mathfrak{gl}(V)_{\mathcal{P}}$. We denote by $X^* \in \mathfrak{gl}(V)_{\mathcal{P}}$ the adjoint of X defined by:

$$(41) \quad \forall v, w \in V_{\mathcal{P}}, B(Xv, w) = B(v, X^*w).$$

We have:

$$(42) \quad (X^*)^* = X.$$

Let $v \in V$, we denote by $B^{\#}(v)$ the element of V^* such that for any $w \in V$, $B^{\#}(v)(w) = B(v, w)$. This defines an isomorphism $B^{\#} : V \rightarrow V^*$. Moreover for $X \in \mathfrak{gl}(V)$ non zero and homogenous we denote by tX the endomorphism of V^* such that for any $\phi \in V^*$ non zero and homogenous and any $v \in V$, ${}^tX(\phi)(v) = (-1)^{p(X)p(f)}\phi(Xv)$. Then:

$$(43) \quad X^* = (B^{\#})^{-1} {}^tX B^{\#}.$$

We denote by $GL(V)_{\mathcal{P}}$ the group of invertible elements of $\mathfrak{gl}(V)_{\mathcal{P}}$. Since $GL(V)_{\mathcal{P}} \subset \mathfrak{gl}(V)_{\mathcal{P}}$, the definition of X^* is meaningful for $X = g \in GL(V)_{\mathcal{P}}$. For $g \in GL(V)_{\mathcal{P}}$ we have from (43): $\text{Ber}(g^*) = \text{Ber}(g)$ and $\text{Ber}_{(1,0)}(g^*) = \text{Ber}_{(1,0)}(g)$.

We put:

$$SpO(V)_{\mathcal{P}} = \{g \in GL(V)_{\mathcal{P}} / g^* = g^{-1}\}.$$

From the multiplicative property of Ber we get for $g \in SpO(V)_{\mathcal{P}}$:

$$(44) \quad \text{Ber}(g) = \text{Ber}_{(1,0)}(g) = \det(\mathbf{b}(g)|_{V_1}) = \pm 1$$

Proposition 2.1. *Let \mathcal{P} be a near superalgebra. Let $X \in \mathfrak{spo}(V \otimes \mathbb{C})_{\mathcal{P}}(\mathcal{V}^+ \times \mathfrak{i} \mathfrak{spo}(V))$ and $g \in GL(V)_{\mathcal{P}}$, then:*

$$(45) \quad \text{Spf}(g^* X g) = \text{Ber}_{(1,0)}^{-1}(g) \text{Spf}(X).$$

In particular, for $g \in SpO(V)_{\mathcal{P}}$, we have;

$$(46) \quad \text{Spf}(g^{-1} X g) = \det(\mathbf{b}(g)|_{V_1}) \text{Spf}(X).$$

Proof. We have by the formula of change of coordinates (cf. [Ber87]):

$$(47) \quad \begin{aligned} \text{Spf}(g^* X g) &= \int_V d_V(v) \exp \left(-\frac{1}{2} B(v, g^* X g v) \right) \\ &= \int_V d_V(v) \exp \left(-\frac{1}{2} B(gv, X g v) \right) \\ &= \int_V d_V(v) \text{Ber}_{(1,0)}^{-1}(g) \exp \left(-\frac{1}{2} B(v, X v) \right) \\ &= \text{Ber}_{(1,0)}^{-1}(g) \text{Spf}(X). \end{aligned}$$

Now, assume that $g \in SpO(V)_{\mathcal{P}}$. Then, by definition, $g^* = g^{-1}$ and formula (46) follows from (44). □

2.4. Action of differential operators. Let V be a symplectic finite dimensional supervector space. We assume that $\dim(V_1)$ is even.

Let $X \in \mathfrak{spo}(V)$ be homogeneous. We denote by ∂_X the derivation of $\mathcal{C}_{\mathfrak{spo}(V)}^{\infty}(\mathfrak{spo}(V))$ such that for any homogeneous $\psi \in \mathfrak{spo}(V_0)^*$:

$$(48) \quad \partial_X \psi = (-1)^{p(X)p(\psi)} \psi(X).$$

The mapping $X \mapsto \partial_X$ extends to an isomorphism between $S(\mathfrak{spo}(V))$ and the superalgebra of differential operators with constant coefficients on $\mathfrak{spo}(V)$.

Let $D \in S(\mathfrak{spo}(V))$. Since $\check{\mu}$ is linear and even on $\mathfrak{spo}(V)$, we have (cf. formula (26) for definition of $\check{\mu}$) for any $X \in \mathfrak{spo}_{\mathcal{P}}(V_0)$ and $v \in V_{\mathcal{P}}$ where \mathcal{P} is a near superalgebra:

$$(49) \quad (\partial_D \exp(\mu))(X, v) = \check{\mu}(D)(v) \exp(\mu)(X, v).$$

Moreover we recall that since Spf is defined by a Gaussian integral all its derivatives are determined by derivation “under the integral”. Thus, if $X \in \mathfrak{spo}(V)_{\mathcal{P}}(\mathcal{V}^+)$:

$$(50) \quad \begin{aligned} \partial_D \int_V d_V(v) \exp(\mu(X, v)) &= \int_V d_V(v) \partial_D \exp(\mu(X, v)) \\ &= \int_V d_V(v) \check{\mu}(D)(v) \exp(\mu(X, v)). \end{aligned}$$

It follows:

Proposition 2.2. *Let $D \in \ker(\check{\mu}) \subset S(\mathfrak{spo}(V))$. Then:*

$$(51) \quad \partial_D \text{Spf} = 0.$$

Let $X, Y \in \mathfrak{spo}(V)$. We put

$$(52) \quad K(X, Y) = \text{str}(XY).$$

It defines a non degenerate symmetric bilinear form on $\mathfrak{spo}(V)$.

Let $(X_i)_{i \in I}$ be a basis of $\mathfrak{spo}(V)$ and $(X'_i)_{i \in I}$ the basis of $\mathfrak{spo}(V)$ such that $K(X_i, X'_j) = \delta_i^j$ (δ_i^j is the Dirac symbol). We put:

$$(53) \quad \square_K = \sum_{i \in I} \partial_{X'_i} \partial_{X_i} \in S^2(\mathfrak{spo}(V)).$$

It is an homogeneous differential operator of degree 2 on $\mathfrak{spo}(V)$.

As a corollary of the above proposition we obtain:

Corollary 2.1. *Let $D \in \bigoplus_{k \in \mathbb{N}^*} S^k(\mathfrak{spo}(V))^{\mathfrak{spo}(V)}$. Thus ∂_D is an $\mathfrak{spo}(V)$ -invariant differential operator on $\mathfrak{spo}(V)$ with constant coefficients and zero scalar term.*

If $\dim(V_0) > 0$:

$$(54) \quad \partial_D \text{Spf} = 0.$$

If $V = V_1$:

$$(55) \quad \deg(D) \neq \frac{\dim(V)}{2} \text{ or } D \in \left(S^{\frac{\dim(V)}{2}}(\mathfrak{so}(V)) \right)^{O(V)} \Rightarrow \partial_D \text{Spf} = 0.$$

In particular, in all cases:

$$(56) \quad \square_K \text{Spf} = 0.$$

Proof. We will use the following lemma:

Lemma 2.1. *For $k \geq 1$, if $\dim(V_0) > 0$ or $V = V_1$ and $k \neq \dim(V)$:*

$$(57) \quad S^k(V^*)^{\mathfrak{spo}(V)} = \{0\}.$$

If $V = V_1$ and $k = \dim(V)$, we have:

$$(58) \quad S^k(V^*)^{\mathfrak{so}(V)} = \Lambda^k(V^*).$$

Proof. Assume that $\dim(V_0) > 0$. Let \mathcal{P} be a near superalgebra. Let $SpSO(V)_{\mathcal{P}}$ be the connected component of $SpO(V)_{\mathcal{P}}$. Since for $v \in V_0 \setminus \{0\}$, $SpSO(V)_{\mathcal{P}}v = V_{\mathcal{P}}$, the invariant polynomials are constants and equality (57) follows.

In case $V = V_1$ cf. [Wey46].

□

Let $D \in S^k(\mathfrak{spo}(V))^{\mathfrak{spo}(V)}$ ($k > 0$), then $\check{\mu}(D) \in S^{2k}(V^*)^{\mathfrak{spo}(V)}$.

Assume that $\dim(V_0) > 0$ or $V = V_1$ and $k \neq \frac{\dim(V)}{2}$, then by Lemma 2.1 we have $\check{\mu}(D) = 0$.

Now assume that $V = V_1$ and $k = \frac{\dim(V)}{2}$. Then $\check{\mu}(D) \in \Lambda^{\dim(V)}(V^*)$. In this case, if $\check{\mu}(D)$ is $O(V)$ invariant, $\check{\mu}(D) = 0$.

Now, the corollary follows from the proposition.

□

2.5. Taylor formula. We still assume that V is a symplectic finite dimensional super-vector space with $n = \dim(V_1)$ even.

2.5.1. *General case.* Let $(P_k)_{k \in \mathbb{N}}$ be an homogeneous (for parity) basis of $S(V^*)$. We assume that P_k is also homogeneous in degree as a polynomial.

We define $c_k(X)$ as the coefficient of P_k in the expansion of \exp :

$$(59) \quad \exp(\mu(X, v)) = \sum_{k \in \mathbb{N}} P_k(v) c_k(X),$$

$c_k \in S(\mathfrak{spo}(V)^*)$. Then we define $\tilde{c}_k \in \mathcal{C}_{\mathfrak{spo}(V)}^\omega(\mathcal{V}^+)$ by

$$(60) \quad \tilde{c}_k(X) = \int_V d_V(v) P_k(v) \exp(\mu(X, v)).$$

We recall that $\Xi : \bigoplus_{k \in \mathbb{N}} S^{2k}(V^*) \rightarrow \bigoplus_{k \in \mathbb{N}} S^k(\mathfrak{spo}(V))$ was defined in 1.11.

We put

$$(61) \quad \partial_k = \partial_{\Xi(P_k)}.$$

We have:

Lemma 2.2. *For any near superalgebra \mathcal{P} and any $X \in \mathfrak{spo}(V)_{\mathcal{P}}(\mathcal{V}^+)$:*

$$(62) \quad \tilde{c}_k(X) = (\partial_k \text{Spf})(X).$$

Proof. We use formula (50) to show:

$$(63) \quad \begin{aligned} \partial_k \int_V d_V(v) \exp(\mu(X, v)) &= \int_V d_V(v) \check{\mu}(\Xi(P_k))(v) \exp(\mu(X, v)) \\ &= \int_V d_V(v) P_k(v) \exp(\mu(X, v)) \\ &= \tilde{c}_k(X). \end{aligned}$$

□

Proposition 2.3. *Let \mathcal{P} be a near superalgebra. Let $X, Y \in \mathfrak{spo}(V)_{\mathcal{P}}$ such that $\mathbf{b}(Y) \in \mathfrak{so}(V_1)$, and $\mathbf{b}(X) \in \mathcal{U}^+$. In this case $\mathbf{b}(X + Y) \in \mathcal{U}^+$. Taylor's formula for Spf reads:*

$$(64) \quad \text{Spf}(X + Y) = \sum_{k \in \mathbb{N}} (-1)^{p(P_k)} c_k(Y) \tilde{c}_k(X).$$

The sum converges as an analytic function in X .

Proof. We have

$$\exp(\mu(X + Y, v)) = \exp(\mu(Y, v)) \exp(\mu(X, v)).$$

Then, we expand $\exp(\mu(Y, v))$ by formula (59):

$$\exp(\mu(Y, v)) = \sum_{k \in \mathbb{N}} P_k(v) c_k(Y) = \sum_{k \in \mathbb{N}} (-1)^{p(P_k)} c_k(Y) P_k(v);$$

and integrate against $d_V(v)$ (since $\dim(V_1)$ is even, this operation is even and thus commute with multiplication by $c_k(Y)$ on the left).

□

2.5.2. *Case* $V = V_1$. In the particular case where $V = V_1$ and Spf is the ordinary pfaffian Pfaff, we have the following simplification. Since here the situation is purely algebraic, we can work with \mathbb{C} as ground field. We fix an oriented orthonormal basis (f_1, \dots, f_n) of V . Let (ξ^1, \dots, ξ^n) be its dual basis. Then $(\xi^J)_{J \in \{0,1\}^n}$ is a basis of $S(V^*)$. We define as above c_J , \tilde{c}_J and ∂_J .

For $J \in \{0,1\}^n$, we put

$$(65) \quad V_J = \mathbb{C}j_1f_1 + \dots + \mathbb{C}j_nf_n.$$

For $J = (j_1, \dots, j_n) \in \{0,1\}^n$ we denote by $J' = (j'_1, \dots, j'_n) \in \{0,1\}^n$ its complementary: $j_i + j'_i = 1$. We have $V = V_J \oplus V_{J'}$. We denote by $p_J : V \rightarrow V_J$ the projection of V onto V_J with $\ker(p_J) = V_{J'}$.

For $|J|$ odd and $Y \in \mathfrak{so}(V)_{\mathcal{P}}$, we have $c_J(Y) = 0$. We now consider the case $|J|$ even.

Since (f_1, \dots, f_n) is an orthonormal oriented basis of V the non-degenerate symmetric bilinear form on V_1 restricts to a non-degenerate symmetric bilinear form on V_J . Let (ξ^1, \dots, ξ^n) be the dual basis of (f_1, \dots, f_n) . We give to V_J the orientation defined by ξ^J . Let $1 \leq j_1 < \dots < j_r \leq n$ such that $\xi^J = \xi^{j_1} \dots \xi^{j_r}$. Then $(f_{j_1}, \dots, f_{j_r})$ is an orthonormal oriented basis of V_J .

Let $Y \in \mathfrak{so}(V_1)$. We put:

$$(66) \quad \begin{aligned} Y_J : V_J &\rightarrow V_J \\ v &\mapsto Y_J(v) = p_J(Y(v)). \end{aligned}$$

We have $Y_J \in \mathfrak{so}(V_J)$. The matrix of Y_J in the basis $(f_{j_1}, \dots, f_{j_r})$ is obtained from the matrix of Y in the basis (f_1, \dots, f_n) as the submatrix corresponding of rows and columns (j_1, \dots, j_r) .

We have:

$$(67) \quad c_J(Y) = (-1)^{\frac{|J|(|J|-1)}{2}} \text{Pfaff}(Y_J).$$

For $J = (1, \dots, 1)$ it is the definition of Pfaff, and in the other cases it follows (see [MQ86]) by evaluating $\exp(\mu(X, v))$ at $\xi^{j'_1} = \dots = \xi^{j'_{n-r}} = 0$.

We define $\epsilon(J, J') \in \{-1, 1\}$ by the formula:

$$(68) \quad \epsilon(J, J') \xi^J \xi^{J'} = \xi^1 \dots \xi^n;$$

it is the signature of the permutation $(1, \dots, n) \mapsto (j_1, \dots, j_r, j'_1, \dots, j'_{n-r})$.

We obtain for $Y \in \mathfrak{so}(V_1)$:

$$(69) \quad \begin{aligned} c_{J'}(Y) &= (-1)^{\frac{n(n-1)}{2}} \int_V d_V(v) (\xi^1 \dots \xi^n)(v) c_{J'}(Y) \\ &= (-1)^{\frac{n(n-1)}{2}} \int_V d_V(v) (\epsilon(J, J') \xi^J \xi^{J'})(v) c_{J'}(Y) \\ &= (-1)^{\frac{n(n-1)}{2}} \epsilon(J, J') \int_V d_V(v) \xi^J(v) \exp(\mu(Y, v)) \\ &= (-1)^{\frac{n(n-1)}{2}} \epsilon(J, J') \tilde{c}_J(Y). \end{aligned}$$

Since for $|J|$ even: $(-1)^{\frac{n(n-1)}{2} + \frac{|J|(|J|-1)}{2} + \frac{|J'|(|J'|-1)}{2}} = 1 = (-1)^{|J|}$, formula (64) reads (cf. [MQ86, Ste90]):

$$(70) \quad \text{Pfaff}(X + Y) = \sum_{J \in \{0,1\}^n / |J| \text{ even}} \epsilon(J, J') \text{Pfaff}(X_J) \text{Pfaff}(Y_{J'}).$$

2.5.3. *Case* $X \in (\mathfrak{spo}(V)_0)_{\mathcal{P}}$ and $Y \in (\mathfrak{spo}(V)_1)_{\mathcal{P}}$. We fix a symplectic oriented basis (e_i, f_j) of V . Let (x^i, ξ^j) its dual basis. Let $I = (i_1, \dots, i_m) \in \mathbb{N}^m$ be a multiindices. We put:

$$(71) \quad x^I = (x^1)^{i_1} \dots (x^m)^{i_m}.$$

Then:

$$(\xi^J x^I)_{(I,J) \in \mathbb{N}^m \times \{0,1\}^n},$$

is a basis of $S(V^*)$.

For $I = (i_1, \dots, i_m) \in \mathbb{N}^m$ we put $|I| = i_1 + \dots + i_m$ and $I! = i_1! \dots i_m!$. Moreover we put:

$$(72) \quad \frac{\partial^{|I|}}{\partial x^I} = \frac{\partial^{|I|}}{(\partial x^1)^{i_1} \dots (\partial x^m)^{i_m}}$$

$$(73) \quad \frac{\partial^{|J|}}{\partial x^J} = \frac{\partial^{|J|}}{(\partial \xi^1)^{j_1} \dots (\partial \xi^n)^{j_n}}.$$

Let \mathcal{P} be a near superalgebra. Let $v \in V \otimes V^*$ be the generic point of V . Let $X \in \mathfrak{spo}(V)_{\mathcal{P}}$. We define $c_{I,J}(X)$ as the coefficient of $\xi^J x^I$ in the Taylor formula:

$$(74) \quad \exp(\mu(X, v)) = \sum_{(I,J) \in \mathbb{N}^m \times \{0,1\}^n} \xi^J x^I c_{I,J}(X).$$

(In particular, for $X \in \mathfrak{spo}(V)_{\mathcal{P}}$ such that $\mathbf{b}(X) \in \mathfrak{so}(V_1)$, since $\mu(X, v)$ is nilpotent, the sum is finite.) It defines $c_{I,J} \in S^{\frac{|I|+|J|}{2}}(\mathfrak{spo}(V)^*)$. We put:

$$(75) \quad \tilde{c}_{I,J}(X) = \int_V d_V(v) (\xi^J x^I)(v) \exp(\mu(X, v)).$$

and for $A \in \mathfrak{sp}(V_0)_{\mathcal{P}}$:

$$(76) \quad \tilde{c}_I(A) = \int_{V_0} d_{V_0}(v_0) (x^I)(v_0) \exp\left(\frac{1}{2} B(v_0, Av_0)\right).$$

To avoid confusing notations, in the rest of this paragraph we denote by \mathbf{B} the symplectic form on V . We put:

$$(77) \quad X = \begin{pmatrix} A & 0 \\ 0 & D \end{pmatrix} \text{ and } Y = \begin{pmatrix} 0 & B \\ C & 0 \end{pmatrix}.$$

with $A \in \mathfrak{sp}(V_0)_{\mathcal{P}}$, $D \in \mathfrak{so}(V_1)_{\mathcal{P}}$, $B \in \text{Hom}(V_1, V_0)_{\mathcal{P}}$ and $C = -B^* \in \text{Hom}(V_0, V_1)_{\mathcal{P}}$. Where B^* is defined by:

$$\forall v \in V_0 \otimes \mathcal{P}_0, \forall w \in V_1 \otimes \mathcal{P}_1, \mathbf{B}(B^*v, w) = \mathbf{B}(v, Bw)$$

Let v_0 be the generic point of V_0 , v_1 be the generic point of V_1 and $v = v_0 + v_1$ be the generic point of V . We have:

$$(78) \quad \mu(X, v) = -\frac{1}{2} \mathbf{B}(v_0, Av_0) - \frac{1}{2} \mathbf{B}(v_1, Dv_1).$$

Assume that $\mathbf{b}(X) \in \mathcal{V}^+$ that means $\mathbf{b}(A) \in \mathcal{U}^+$. Then, with the notations of the preceding subsection, for $|J|$ even, we have $(-1)^{\frac{|J'|(|J'|-1)}{2} + \frac{n(n-1)}{2}} = (-1)^{\frac{|J|(|J|-1)}{2}}$, and so:

$$\begin{aligned} \tilde{c}_{I,J}(X) &= \int_{V_0} d_{V_0}(v_0) x^I(v_0) \exp\left(-\frac{1}{2} \mathbf{B}(v_0, Av_0)\right) \int_{V_1} d_{V_1}(v_1) \xi^J(v_1) \exp\left(-\frac{1}{2} \mathbf{B}(v_1, Dv_1)\right) \\ &= (-1)^{\frac{|J|(|J|-1)}{2}} \epsilon(J, J') \tilde{c}_I(A) \text{Pfaff}(D_{J'}). \end{aligned} \quad (79)$$

Now, we explicit $c_{I,J}(Y)$. Let us introduce some notations.
Since $B = -C^*$ we have:

$$\begin{aligned} \mu(Y, v) &= -\frac{1}{2} (\mathbf{B}(v_0, Bv_1) + \mathbf{B}(v_1, Cv_0)) \\ &= -\frac{1}{2} (\mathbf{B}(B^*v_0, v_1) + \mathbf{B}(v_1, Cv_0)) \\ &= -\frac{1}{2} (-\mathbf{B}(Cv_0, v_1) + \mathbf{B}(v_1, Cv_0)) \\ &= -\mathbf{B}(v_1, Cv_0). \end{aligned}$$

Let $(I, J) \in \mathbb{N}^m \times \{0, 1\}^n$. We denote by $C_{J,I}$ the $|J| \times |I|$ matrix obtained from C by keeping j_k times the k -th line of C (in other words we keep the lines (j_1, \dots, j_r)) and i_k times the k -th column of C .

Example: Assume that $(m, n) = (3, 4)$. Let:

$$C = \begin{pmatrix} \alpha_1 & \alpha_2 & \alpha_3 \\ \beta_1 & \beta_2 & \beta_3 \\ \gamma_1 & \gamma_2 & \gamma_3 \\ \delta_1 & \delta_2 & \delta_3 \end{pmatrix}.$$

Let $I = (2, 0, 1)$ and $J = (0, 1, 1, 1)$. Then:

$$C_{J,I} = \begin{pmatrix} \beta_1 & \beta_1 & \beta_3 \\ \gamma_1 & \gamma_1 & \gamma_3 \\ \delta_1 & \delta_1 & \delta_3 \end{pmatrix}.$$

Let $r \in \mathbb{N}$. We denote by \mathfrak{S}_r the group of permutations of $\{1, \dots, r\}$. We denote by ϕ_r the r -multilinear form on the $r \times r$ matrix antisymmetric in the lines, symmetric in the columns defined for $M = (a_{i,j})_{1 \leq i, j \leq r}$ with $a_{i,j} \in \mathcal{P}_1$ by:

$$(80) \quad \phi_r(M) = \sum_{\sigma \in \mathfrak{S}_r} a_{1,\sigma(1)} \dots a_{r,\sigma(r)}$$

In the sequel we put for $C \in \text{Hom}(V_0, V_1) \otimes \mathcal{P}_1$ and $I, J \in \mathbb{N}^m \times \{0, 1\}^n$:

$$(81) \quad c_{I,J}(C) = \begin{cases} 0 & \text{if } |I| \neq |J|; \\ (-1)^{\frac{|J|(|J|-1)}{2}} \phi_{|J|}(C_{J,I}) & \text{if } |I| = |J|. \end{cases}$$

With this notations we have:

$$(82) \quad c_{I,J}(Y) = c_{I,J}(C).$$

Example: We take the preceding example with $(m, n) = (3, 4)$, $Y = \begin{pmatrix} 0 & B \\ C & 0 \end{pmatrix}$ with C as above, $I = (2, 0, 1)$ and $J = (0, 1, 1, 1)$. Then:

$$c_{I,J}(Y) = c_{I,J}(C) = -\phi_3(C_{J,I}) = -2(\beta_1\gamma_1\delta_3 + \beta_1\gamma_3\delta_1 + \beta_3\gamma_1\delta_1)$$

Now formula (64) gives:

Proposition 2.4. *Let \mathcal{P} be any near superalgebra. Let $\begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \mathfrak{spo}(V)_{\mathcal{P}}(\mathcal{V}^+)$, then:*

$$(83) \quad \text{Spf} \begin{pmatrix} A & B \\ C & D \end{pmatrix} = \sum_{(I,J) \in \mathbb{N}^m \times \mathcal{J}_n / |I|=|J| \text{ even}} (-1)^{\frac{|J|(|J|-1)}{2}} \epsilon(J, J') c_{I,J}(C) \text{Pfaff}(D_{J'}) \tilde{c}_I(A).$$

2.6. $\text{Spf}(-X^{-1})$ and Spf^2 .

2.6.1. *Some formulas.* Let W be a supervector space. Let $\mathbf{w} \in (V \otimes W^*)_{\mathbf{0}}$. Let \mathcal{P} be any near superalgebra. For $w \in W_{\mathcal{P}}$, $\mathbf{w}(w)$ belongs to $V_{\mathcal{P}}$ and $v \mapsto B(v, \mathbf{w}(w))$ is linear on V while $v \mapsto B(v, Xv)$ is quadratic. Thus, as in proof of theorem 2.1 (cf. section 2.1), for any near superalgebra \mathcal{P} we define an analytic function $\psi_{\mathcal{P}}$ on $\mathfrak{spo}(V)_{\mathcal{P}}(\mathcal{V}^+) \times W_{\mathcal{P}} \times \mathbb{C}$ by the formula $((X, w, \lambda) \in \mathfrak{spo}(V)_{\mathcal{P}}(\mathcal{V}^+) \times W_{\mathcal{P}} \times \mathbb{C})$:

$$(84) \quad \psi_{\mathcal{P}}(X, w, \lambda) = \int_V d_V(v) \exp(\mu(X, v) + \lambda B(v, \mathbf{w}(w))).$$

Then, as in the proof of theorem 2.1, we can prove that there is a function ψ on $\mathfrak{spo}(V) \times W \times \mathbb{C}$ defined on $\mathcal{V}^+ \times W_{\mathbf{0}} \times \mathbb{C}$ such that for any near superalgebra \mathcal{P} and $\forall (X, w, \lambda) \in \mathfrak{spo}(V)_{\mathcal{P}}(\mathcal{V}^+) \times W_{\mathcal{P}} \times \mathbb{C}$, $\psi(X, w, \lambda) = \psi_{\mathcal{P}}(X, w, \lambda)$.

Now, we put:

$$\text{Spf}_{\lambda}^{\mathbf{w}}(X, w) = \psi(X, w, \lambda).$$

Thus, for $\lambda \in \mathbb{C}$, $\text{Spf}_{\lambda}^{\mathbf{w}} \in \mathcal{C}_{\mathfrak{spo}(V) \times W}^{\infty}(\mathcal{V}^+ \times W_{\mathbf{0}})$.

Lemma 2.3. (cf. [MQ86] for the case $V = V_1$.) *Let $X \in \mathfrak{spo}(V)_{\mathcal{P}}(\mathcal{V}^+)$ be invertible (since $\mathbf{b}(X)|_{V_0}$ is already invertible, it means $\mathbf{b}(X)|_{V_1}$ invertible) and $w \in W_{\mathcal{P}}$. We have:*

$$(85) \quad \text{Spf}_{\lambda}^{\mathbf{w}}(X, w) = \text{Spf}(X) \exp\left(\frac{-\lambda^2}{2} B(\mathbf{w}(w), X^{-1}\mathbf{w}(w))\right)$$

Proof. Since $X^* = -X$, we have:

$$(86) \quad \begin{aligned} \mu(X, v) + \lambda B(v, \mathbf{w}(w)) &= -\frac{1}{2} B(v, Xv) + \lambda B(v, \mathbf{w}(w)) \\ &= -\frac{1}{2} B(v - \lambda X^{-1}\mathbf{w}(w), X(v - \lambda X^{-1}\mathbf{w}(w))) \\ &\quad + \frac{\lambda^2}{2} B(X^{-1}\mathbf{w}(w), \mathbf{w}(w)). \end{aligned}$$

We put:

$$(87) \quad \phi(X, \lambda, w) = \int_V d_V(v) \exp\left(-\frac{1}{2} B(v - \lambda X^{-1}\mathbf{w}(w), X(v - \lambda X^{-1}\mathbf{w}(w)))\right).$$

It is an analytic function on $\mathcal{V}^+ \times \mathbb{C} \times V$. Since $d_V(v)$ is invariant by translations, we have on $\mathcal{V}^+ \times \mathbb{R} \times V$: $\phi(X, \lambda, w) = \text{Spf}(X)$. By uniqueness of analytic continuation, it follows that for any $\lambda \in \mathbb{C}$, $\phi(X, \lambda, w) = \text{Spf}(X)$. \square

Applying this lemma for various particular values of W and \mathbf{w} we will obtain some useful formulas. First, we take W to be a supervector space isomorphic to V and $\mathbf{w} \in V \otimes W^*$ be an isomorphism $W \rightarrow V$.

Let $D \in S(V)$. It defines a differential operator ∂_D on V . We put:

$$(88) \quad \bar{c}_D(X) = \left(\partial_D \exp(\mu) \right)(X, 0).$$

and for $P \in S(V^*)$ and $X \in \mathfrak{spo}(V)_{\mathcal{P}}(\mathcal{V}^+)$:

$$(89) \quad \tilde{c}_P(X) = \int_V d_V(v) P(v) \exp(\mu(X, v)).$$

We recall (cf. section 2.3) that $B^\# : V \rightarrow V^*$ is an isomorphism. It extends to an isomorphism of algebras $B^\# : S(V) \rightarrow S(V^*)$. We put for $D \in S(V)$:

$$(90) \quad D^\# = B^\#(D).$$

Let $v \in V \otimes V^*$ be the generic point of V . For $D \in S^k(V)$ we have:

$$(91) \quad \partial_D v^k = k! D;$$

and:

$$(92) \quad \partial_{\mathbf{w}^{-1}(D)}(B(v, \mathbf{w})^k) = (-1)^k k! D^\#(v).$$

Example: Let $(e_1, \dots, e_m, f_1, \dots, f_n)$ be an homogeneous basis of V . Let $(x^1, \dots, x^m, \xi^1, \dots, \xi^n)$ be its dual basis. Let $(I, J) \in \mathbb{N}^m \times \{0, 1\}^n$. We put $D_{I,J} = e_1^{i_1} \dots e_m^{i_m} f_1^{j_1} \dots f_n^{j_n}$. We have for $X \in \mathfrak{spo}(V)_{\mathcal{P}}$ ($\partial_{e_i} = \frac{\partial}{\partial x^i}$ and $\partial_{f_j} = -\frac{\partial}{\partial \xi^j}$):

$$\begin{aligned} \partial_{D_{I,J}} \xi^J x^I &= (-1)^{|J|} \left(\frac{\partial}{\partial x^1} \right)^{i_1} \dots \left(\frac{\partial}{\partial x^m} \right)^{i_m} \left(\frac{\partial}{\partial \xi^1} \right)^{j_1} \dots \left(\frac{\partial}{\partial \xi^n} \right)^{j_n} \xi^J x^I \\ &= I! (-1)^{\frac{|J|(|J|+1)}{2}} \end{aligned}$$

Then:

$$\begin{aligned} \bar{c}_{D_{I,J}}(X) &= I! (-1)^{\frac{|J|(|J|+1)}{2}} c_{I,J}(X) \\ \text{and } \exp(\mu(X, v)) &= \sum_{(I,J) \in \mathbb{N}^m \times \{0,1\}^n} \frac{\xi^J x^I}{I! (-1)^{\frac{|J|(|J|+1)}{2}}} \bar{c}_{D_{I,J}}(X). \end{aligned}$$

We obtain:

Corollary 2.2. *For any $X \in \mathfrak{spo}(V)_{\mathcal{P}}(\mathcal{V}^+)$ invertible, and $D \in S^{2k}(V)$ (if $D \in S^{2k+1}(V)$ $\bar{c}_{D^\#}(X) = c_D(-X^{-1}) = 0$):*

$$(93) \quad \tilde{c}_{D^\#}(X) = \bar{c}_D(X^{-1}) \text{Spf}(X).$$

and for $\Re(\lambda^2) < 0$ (in this case $\lambda^2 X^{-1} \in \mathfrak{spo}(V)_{\mathcal{P}}(\mathcal{V}^+) \times \mathfrak{ispo}(V)_{\mathcal{P}}$):

$$(94) \quad (-\lambda^2)^{\frac{m-n-2k}{2}} (-1)^k \bar{c}_D(X) \bar{c}_{D^\#}(X) = \tilde{c}_{D^\#}(\lambda^2 X^{-1}) \text{Spf}(X).$$

Remark: We point out that in case $V = V_0$ formula (93) is Wick formula (cf. for example [GJ81]).

Proof. Let $v \in V \otimes V^*$ be the generic point of V . Let $D \in S^{2k}(V)$. Then:

$$\partial_{\mathbf{w}^{-1}(D)} \exp(\lambda B(v, \mathbf{w})) = (-\lambda)^{2k} D^\#(v) \exp(\lambda B(v, \mathbf{w})).$$

Now, we apply $\frac{1}{\lambda^{2k}} \partial_{\mathbf{w}^{-1}(D)}$ to equality (85) and then, we take the value at $(X, 0)$. Since, $\bar{c}_D(\lambda^2 X^{-1}) = \lambda^{2k} \bar{c}_D(X^{-1})$, formula (93) follows.

For the second formula, we need an auxiliary result.

Consider the application:

$$\phi \mapsto \mathcal{F}_\lambda(\phi) = \int_W d_W(w) \int_V d_V(v) \phi(v) \exp(\lambda B(\mathbf{w}(w), v)).$$

It is defined for $\phi \in \mathcal{C}_V^\infty(V_0)$ such that for any $w \in W_{\mathcal{P}}$, $v \mapsto \phi(v) \exp(\lambda B(\mathbf{w}(w), v))$ and all its derivatives is rapidly decreasing on V . It is linear and $\phi(0) = 0$ implies that $\mathcal{F}_\lambda(\phi) = 0$. Thus, there is $K_\lambda \in \mathbb{C}$ such that $\mathcal{F}_\lambda(\phi) = K_\lambda \phi(0)$. To find K_λ it is enough to consider a particular ϕ . For example $\phi(v) = \exp(\mu(X, v))$ ($\phi = \exp(\check{\mu}(X))$) for some $X \in \mathfrak{spo}(V)_{\mathcal{P}}(\mathcal{V}^+)$ fixed. In this case formula (85) shows that:

$$\mathcal{F}_\lambda(\exp(\check{\mu}(X))) = \int_V d_V(v) \exp(\mu(X, v)) \int_W d_W(w) \exp\left(\frac{-\lambda^2}{2} B(\mathbf{w}(w), X^{-1} \mathbf{w}(w))\right).$$

hypothesis $\Re(\lambda^2) < 0$ implies that $\lambda^2 X^{-1} \in \mathfrak{spo}(V)_{\mathcal{P}}(\mathcal{V}^+)$ and thus, the above integral converges. More precisely, since $\exp(\mu(X, 0)) = 1$, we have:

$$(95) \quad K_\lambda = \mathcal{F}_\lambda(\exp(\check{\mu}(X))) = \text{Spf}(X) \text{Spf}(\lambda^2 X^{-1}).$$

Since it does not depends on X it is enough to take

$$X = \left(\begin{array}{cc|cc} J_2 & 0 & 0 & 0 \\ & \ddots & & \\ 0 & & J_2 & 0 \\ \hline 0 & 0 & J_2 & 0 \\ & \ddots & & \\ 0 & 0 & 0 & J_2 \end{array} \right)$$

where $J_2 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$. We obtain:

$$(96) \quad K_\lambda = (-\lambda^2)^{\frac{n-m}{2}}.$$

Now, we multiply both sides of (85) by $D^\#(\mathbf{w}(w))$ and then integrate on W against $d_W(w)$.

For the left hand side we have (the first equality is obtained by integration by parts):

$$\begin{aligned}
& \int_W d_W(w) D^\#(w) \int_V d_V(v) \exp(\mu(X, v) + \lambda B(v, \mathbf{w}(w))) \\
&= (-\lambda)^{-2k} \int_W d_W(w) \int_V d_V(v) \left(\partial_D \exp(\mu) \right) (X, v) \exp(\lambda B(v, \mathbf{w}(w))) \\
&= (-\lambda)^{-2k} \mathcal{F}_\lambda \left(\partial_D \exp(\check{\mu}(X)) \right) \\
&= (-\lambda)^{-2k} (-\lambda^2)^{\frac{n-m}{2}} \left(\partial_D \exp(\check{\mu}(X)) \right) (0) \\
&= (-\lambda^2)^{\frac{n-m-2k}{2}} (-1)^k \bar{c}_D(X).
\end{aligned}$$

□

As a corollary of equation (85) we also have:

Corollary 2.3. *Let \mathcal{P} be a near superalgebra.*

Let $A \in \mathfrak{sp}(V_0)_{\mathcal{P}}(\mathcal{U}^+)$, $B \in \text{Hom}(V_1, V_0) \otimes \mathcal{P}_1$ and $C \in \text{Hom}(V_0, V_1) \otimes \mathcal{P}_1$ such that $\begin{pmatrix} A & B \\ C & 0 \end{pmatrix} \in \mathfrak{spo}(V)_{\mathcal{P}}$. Then for any $J \in \{0, 1\}^n$ even:

$$\begin{aligned}
(97) \quad \sum_{I \in \mathbb{N}^m / |I|=|J|} (-i)^{|J|} \tilde{c}_I(A) c_{I,J}(C) &= \frac{1}{\sqrt{\det(A)}} c_J(CA^{-1}B) \\
&= \frac{(-1)^{\frac{|J|(|J|-1)}{2}}}{\sqrt{\det(A)}} \text{Pfaff}((CA^{-1}B)_J).
\end{aligned}$$

Proof. In this proof to avoid confusing notations, as in subsection 2.5.3 we denote by \mathbf{B} the symplectic form on V .

For $v \in (V_0)_{\mathcal{P}}$ and $w \in V_1$. we have:

$$(98) \quad \mathbf{B}(v, Bw) = -\mathbf{B}(Cv, w) = \mathbf{B}(w, Cv).$$

Thus if $\mathcal{P} = S(V^*)$, $v = \sum_i e_i x^i$ (resp. $w = \sum_j f_j \xi^j$) is the generic point of V_0 (resp. V_1), the coefficient of $\xi^J x^I$ in $\exp(-\mathbf{B}(v, Bw)) = \exp\left(-\frac{1}{2}(\mathbf{B}(v, Bw) + \mathbf{B}(w, Cv))\right)$ is $c_{I,J}(C)$ (cf. (82)). On the other hand, we have for $w \in (V_1)_{\mathcal{P}}$:

$$(99) \quad \mathbf{B}(Bw, A^{-1}Bw) = -\mathbf{B}(w, CA^{-1}Bw);$$

Now, look at equation (85) with $V = V_0$, $W = V_1$, $X = A$, and:

$$(100) \quad \mathbf{w}(w) = Bw.$$

Then we apply $\left(\frac{\partial}{\partial \xi^n}\right)^{j_n} \dots \left(\frac{\partial}{\partial \xi^1}\right)^{j_1}$ to (85) and taking value at $(X, 0)$. Then equality (97) follows by multiplying each side by $\frac{(-i)^{|J|}}{\lambda^{|J|}}$. □

Similarly, we obtain:

Corollary 2.4. *Let \mathcal{P} be a near superalgebra.*

Let $D \in \mathfrak{so}(V_1)_{\mathcal{P}}$ invertible, $B \in \text{Hom}(V_1, V_0) \otimes \mathcal{P}_1$ and $C \in \text{Hom}(V_0, V_1) \otimes \mathcal{P}_1$ such that $\begin{pmatrix} 0 & B \\ C & D \end{pmatrix} \in \mathfrak{spo}(V)_{\mathcal{P}}$. Then for any $I \in \mathbb{N}^m$:

$$(101) \quad \sum_{J \in \mathcal{J}_n / |I|=|J|} \epsilon(J, J') (-1)^{\frac{|J|(|J|-1)}{2}} \text{Pfaff}(D_{J'}) c_{I,J}(C) = \text{Pfaff}(D) c_I(BD^{-1}C).$$

Proof. Here we apply formula (85) with $V = V_1$, $W = V_0$, $X = D$ and

$$\mathbf{w}(w) = Cw.$$

Now, we apply

$$\frac{1}{I!} \frac{\partial^{i_1}}{\partial y_1^{i_1}} \cdots \frac{\partial^{i_m}}{\partial y_m^{i_m}}$$

to (85) and taking value at $(X, 0)$.

Then, using (67) and (67), equality (97) follows from multiplication of each side by $\frac{(-1)^{\frac{|J|}{2}}}{\lambda^{|J|}}$. \square

2.6.2. Evaluation of Spf^2 .

Proposition 2.5. *We have in $\mathcal{C}_{\mathfrak{spo}(V)}^{\omega}(\mathcal{V}^+)$.*

$$(102) \quad \text{Spf}^2 = \text{Ber}^-.$$

It is equivalent to say that for any near superalgebra \mathcal{P} and any $X \in \mathfrak{spo}(V)_{\mathcal{P}}(\mathcal{V}^+) \times \mathfrak{i} \mathfrak{spo}(V)_{\mathcal{P}}$ we have:

$$(103) \quad \text{Spf}^2(X) = \text{Ber}^-(X).$$

Proof. Proposition 2.1 with $g = X^{-1}$ gives:

$$(104) \quad \text{Spf}(-X^{-1}) = \text{Ber}_{(1,0)}^{-1}(X^{-1}) \text{Spf}(X).$$

Now, since on \mathcal{V}^+ , $\text{Ber}_{(1,0)}^{-1} = \text{Ber}^-$, the result follows from multiplying both sides by $\text{Ber}^-(X) \text{Spf}(X)$ and using formula (95) with $\lambda = \mathfrak{i}$. \square

2.7. Product formulas.

2.7.1. *Case $X \in \mathfrak{spo}(V)_{\mathcal{P}}(\mathcal{U}^+)$.* Let $X = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$. We put $X^* = \begin{pmatrix} A^* & C^* \\ B^* & D^* \end{pmatrix}$. Then, $X \in \mathfrak{spo}(V)_{\mathcal{P}} \Leftrightarrow X^* = -X$. Moreover, $X \in \mathfrak{spo}(V)_{\mathcal{P}}(\mathcal{U}^+)$ implies that A is invertible. It follows that:

$$(105) \quad \begin{aligned} A^* &= -A \\ D^* &= -D \\ C^* &= -B \end{aligned}$$

Thus:

$$(106) \quad \begin{aligned} \begin{pmatrix} A & B \\ C & D \end{pmatrix} &= \begin{pmatrix} 1 & 0 \\ CA^{-1} & 1 \end{pmatrix} \begin{pmatrix} A & 0 \\ 0 & D - CA^{-1}B \end{pmatrix} \begin{pmatrix} 1 & A^{-1}B \\ 0 & 1 \end{pmatrix}, \\ &= \begin{pmatrix} 1 & A^{-1}B \\ 0 & 1 \end{pmatrix}^* \begin{pmatrix} A & 0 \\ 0 & D - CA^{-1}B \end{pmatrix} \begin{pmatrix} 1 & A^{-1}B \\ 0 & 1 \end{pmatrix}. \end{aligned}$$

Since $\text{Ber}_{(1,0)} \begin{pmatrix} 1 & A^{-1}B \\ 0 & 1 \end{pmatrix} = 1$, formula (45) imply

$$(107) \quad \text{Spf} \begin{pmatrix} A & B \\ C & D \end{pmatrix} = \text{Spf} \begin{pmatrix} A & 0 \\ 0 & D - CA^{-1}B \end{pmatrix} = \frac{\text{Pfaff}(D - CA^{-1}B)}{\sqrt{\det(A)}}.$$

We explain the relation with formula (83).

Taylor formula (70) for $\text{Pfaff}(D - CA^{-1}B)$ gives:

$$(108) \quad \text{Pfaff}(D - CA^{-1}B) = \sum_{J \in \{0,1\}^n / |J| \text{ even}} \epsilon(J, J') \text{Pfaff}((-CA^{-1}B)_J) \text{Pfaff}(D_{J'}).$$

Compatibility with formula (83) follows from formula (97) and that for $|J|$ even $(-\mathbf{i})^{|J|} = (-1)^{\frac{|J|(|J|-1)}{2}}$.

2.7.2. *Case $X \in \mathfrak{spo}(V)_{\mathcal{P}}(\mathcal{U}^+)$ and invertible.* As before we put $X = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$ but now we assume moreover that D is invertible. We have:

$$(109) \quad \begin{aligned} \begin{pmatrix} A & B \\ C & D \end{pmatrix} &= \begin{pmatrix} 1 & BD^{-1} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} A - BD^{-1}C & 0 \\ 0 & D \end{pmatrix} \begin{pmatrix} 1 & 0 \\ D^{-1}C & 1 \end{pmatrix}, \\ &= \begin{pmatrix} 1 & 0 \\ D^{-1}C & 1 \end{pmatrix}^* \begin{pmatrix} A - BD^{-1}C & 0 \\ 0 & D \end{pmatrix} \begin{pmatrix} 1 & 0 \\ D^{-1}C & 1 \end{pmatrix}. \end{aligned}$$

Since $\text{Ber}_{(1,0)} \begin{pmatrix} 1 & 0 \\ D^{-1}C & 1 \end{pmatrix} = 1$, formula (45) imply

$$(110) \quad \text{Spf} \begin{pmatrix} A & B \\ C & D \end{pmatrix} = \text{Spf} \begin{pmatrix} A - BD^{-1}C & 0 \\ 0 & D \end{pmatrix} = \frac{\text{Pfaff}(D)}{\sqrt{\det(A - BD^{-1}C)}}.$$

Now, compatibility with formula (83) comes from formula (64) for $\frac{1}{\sqrt{\det(A - BD^{-1}C)}}$ and equation (101).

2.8. Homogeneity.

Corollary 2.5. *The superPfaffian is an homogeneous function of degree $\frac{n-m}{2}$ on $\mathcal{V}^+ \times \mathfrak{spo}(V)$.*

Proof. Let $\lambda > 0$. We denote by $M_\lambda \in GL(V_0) \times GL(V_1)$ the homothecy with ratio λ on V . For any near superalgebra \mathcal{P} , any $X \in \mathfrak{spo}(V \otimes \mathbb{C})_{\mathcal{P}}(\mathcal{V}^+ \times \mathfrak{spo}(V))$, we have:

$$(111) \quad \lambda X = M_{\sqrt{\lambda}} X M_{\sqrt{\lambda}}$$

The corollary follows from proposition 2.1 and the equalities

$$(112) \quad M_{\sqrt{\lambda}}^* = M_{\sqrt{\lambda}} \quad \text{and} \quad \text{Ber}_{(1,0)}(M_{\sqrt{\lambda}}) = (M_{\sqrt{\lambda}}) = \lambda^{\frac{m-n}{2}}.$$

□

3. SUPERPFAFFIAN II : A GENERALIZED FUNCTION

In this section we define Spf as a generalized function on $\mathfrak{spo}(V)$ by the formula:

$$(113) \quad \text{Spf}(X) = i^{\frac{m-n}{2}} \int_V d_V(v) \exp(-\frac{i}{2}B(v, Xv))$$

The meaning of this formula is the following. For any smooth compactly supported distribution t on $\mathfrak{spo}(V)$:

$$(114) \quad \int_{\mathfrak{spo}(V)} t(X) \text{Spf}(X) = i^{\frac{m-n}{2}} \int_V d_V(v) \int_{\mathfrak{spo}(V)} t(X) \exp(-\frac{i}{2}B(v, Xv)).$$

This means:

- (1) we evaluate the integral on $\mathfrak{spo}(V)$;
- (2) the resulting function is rapidly decreasing on V (cf. section 3.1);
- (3) we evaluate the integral on V .

This generalized function on the supermanifold $\mathfrak{spo}(V)$ coincides on \mathcal{V}^+ with the superPfaffian defined in the preceding section (cf. subsection 2 for a proof).

3.1. A well defined generalized function. In this section we prove that formula (113) defines a generalized function on $\mathfrak{sp}(V_0)$, with values in a finite dimensional subspace of $S((\mathfrak{so}(V_1) \oplus \mathfrak{spo}(V)_1)^*)$ in a sense analog to formula (114). In particular this ensures that formula (113) defines generalized function on $\mathfrak{spo}(V)$. The point is to show that $\int_{\mathfrak{spo}(V)} t(X) \exp(-\frac{i}{2}B(v, Xv))$ is rapidly decreasing on V .

Let us precise some notations.

Let $v \in V \otimes V^*$ be the generic point of V . We denote by $\tilde{\mu} \in \mathfrak{spo}(V)^* \otimes S^2(V^*)$ the polynomial of degree 2 on V with values in $\mathfrak{spo}(V)^*$ such that for any near superalgebra \mathcal{P} and any $X \in \mathfrak{spo}(V)_{\mathcal{P}}$:

$$(115) \quad \tilde{\mu}(v)(X) = \mu(X, v).$$

In particular for $u \in V_{\mathcal{P}}$, $\tilde{\mu}(u) \in \mathfrak{spo}(V)_{\mathcal{P}}^*$.

Let ρ be a smooth compactly supported distribution on $\mathfrak{sp}(V_0)$. We denote by $\hat{\rho}$ its Fourier transform. It is a smooth rapidly decreasing function on $\mathfrak{sp}(V_0)^*$ (in sense of Schwartz) which is defined for $f \in \mathfrak{sp}(V_0)^*$ by the formula:

$$(116) \quad \hat{\rho}(f) = \int_{\mathfrak{sp}(V_0)} \rho(X) \exp(-if(X)),$$

We fix $J \in \mathcal{U}^+$. Then $B(v, Jv)$ is a positive definite quadratic form on V_0 . We put for $u \in V_0$, $\|u\| = \sqrt{\frac{1}{2}B(u, Ju)}$.

We fix a norm N' on $\mathfrak{spo}(V)_0$ and denote N the associated norm on $\mathfrak{spo}(V)_0^*$. For $f \in \mathfrak{spo}(V)_0^*$ we have:

$$(117) \quad N(f) = \sup_{Y \in \mathfrak{spo}(V)_0 \setminus \{0\}} \frac{|f(Y)|}{N'(Y)}.$$

Thus we have for $u \in V_0$:

$$(118) \quad N(\tilde{\mu}(u)|_{\mathfrak{sp}(V_0)}) \geq \frac{|\tilde{\mu}(u)(J)|}{N'(J)} = \frac{\|u\|^2}{N'(J)};$$

where $\tilde{\mu}(u)|_{\mathfrak{sp}(V_0)}$ is the restriction of $\tilde{\mu}(v_0)$ to $\mathfrak{sp}(V_0)$.

We put for $X'' \in (\mathfrak{so}(V_1) \oplus \mathfrak{spo}(V)_1)_{\mathcal{P}}$:

$$(119) \quad \phi(X'', v) = \int_{\mathfrak{sp}(V_0)} \rho(X') \exp\left(-\frac{i}{2}B(v, (X' + X'')v)\right)$$

Lemma 3.1. *For any $X'' \in (\mathfrak{so}(V_1) \oplus \mathfrak{spo}(V)_1)_{\mathcal{P}}$, $\phi(X'', v)$ is a well defined rapidly decreasing function on V . Moreover, ϕ is polynomial in X'' .*

Proof. Let $v_0 \in V_0 \otimes V_0^*$ be the generic point of V_0 and $v_1 \in V_1 \otimes V_1^*$ be the generic point of V_1 . As in section 2.1.1 we have $v = v_0 + v_1$. Then:

$$(120) \quad \begin{aligned} B(v, (X' + X'')v) &= B(v_0, X'v_0) + B(v, X''v) \\ &= -2\tilde{\mu}(v_0)(X') + B(v, X''v). \end{aligned}$$

It follows

$$(121) \quad \begin{aligned} \phi(X'', v) &= \int_{\mathfrak{sp}(V_0)} \rho(X') \exp(i\tilde{\mu}(v_0)(X')) \exp\left(-\frac{i}{2}B(v, X''v)\right) \\ &= \hat{\rho}(-\tilde{\mu}(v_0)|_{\mathfrak{sp}(V_0)}) \exp\left(-\frac{i}{2}B(v, X''v)\right). \end{aligned}$$

Since $\mathbf{b}(X'') \in \mathfrak{so}(V_1)$, we have:

$$(122) \quad B(v, X''v) = B(v_1, \mathbf{b}(X'')v_1) + B(v, (X'' - \mathbf{b}(X''))v).$$

Hence, $B(v, X''v)$ is nilpotent and $X'' \mapsto \exp(-\frac{i}{2}B(v, X''v))$ defines a polynomial function on $\mathfrak{so}(V_1) \oplus \mathfrak{spo}(V)_1$ with values in $S(V^*)$. In particular ϕ is polynomial in X'' .

Since $\hat{\rho}$ is rapidly decreasing on $\mathfrak{sp}(V_0)^*$, formula (118) ensures that $\hat{\rho}(-\tilde{\mu}(v_0)|_{\mathfrak{sp}(V_0)})$ is a rapidly decreasing function on V_0 .

Finally, for any $X'' \in (\mathfrak{so}(V_1) \oplus \mathfrak{spo}(V)_1)_{\mathcal{P}}$, $\phi(X'', v)$ is a rapidly decreasing function on V . \square

Now, the integral:

$$(123) \quad h(X'') = \int_V d_V(v) \phi(X'', v) = \int_V d_V(v) \int_{\mathfrak{sp}(V_0)} \rho(X') \exp\left(-\frac{i}{2}B(v, (X' + X'')v)\right)$$

converges and it defines a polynomial function on $\mathfrak{so}(V_1) \oplus \mathfrak{spo}(V)_1$. This means that Spf is a generalized function on $\mathfrak{sp}(V_0)$ with values in $S((\mathfrak{so}(V_1) \oplus \mathfrak{spo}(V)_1)^*)$.

3.1.1. Example: Symplectic 2-dimensional vector space: This is the crucial example. The problem is to show that for a smooth compactly supported distribution ρ on $\mathfrak{sp}(V)$:

$$(124) \quad v \mapsto \int_{\mathfrak{sp}(V)} \rho(X) \exp\left(-\frac{i}{2}B(v, Xv)\right)$$

is a rapidly decreasing function on V .

We use notations and results of sections 1.10.1 and 2.0.1. We denote by $(\mathbf{a}, \mathbf{b}, \mathbf{c})$ the basis of $\mathfrak{sp}(V)^*$ such that $\mathbf{a}(X) = a$, $\mathbf{b}(X) = b$ and $\mathbf{c}(X) = c$.

Thus:

$$(125) \quad \int_{\mathfrak{sp}(V)} \rho(X) \exp(i\mu(X, v)) = \hat{\rho}\left(\frac{1}{2}(cx^2 - 2axy - by^2)\right).$$

Since $\hat{\rho}$ is a rapidly decreasing function on $\mathfrak{sp}(V)^*$, the above function is rapidly decreasing on V .

3.2. Comparison with the analytic version of section 2. In this section we denote by Spf_{an} the analytic superPfaffian defined on $\mathcal{V}^+ \times \mathbf{i}\mathfrak{spo}(V)$ by formula (31) and by Spf_{gene} the generalized superPfaffian defined on $\mathfrak{spo}(V)$ by formula (113).

Since $X + \mathbf{i}Y \in \mathfrak{spo}(V) \times \mathbf{i}\mathcal{V}^-$ is equivalent $\mathbf{i}(X + \mathbf{i}Y) \in \mathcal{V}^+ \times \mathbf{i}\mathfrak{spo}(V)$,

$$(126) \quad (X, \mathbf{i}Y) \mapsto \mathbf{i}^{\frac{m-n}{2}} \text{Spf}_{an}(\mathbf{i}(X + \mathbf{i}Y))$$

is an analytic function on $\mathfrak{spo}(V) \times \mathbf{i}\mathcal{V}^-$. We consider it as an analytic function on the open cone $\mathfrak{spo}(V)_0 \times \mathbf{i}\mathcal{V}^-$ of $\mathfrak{spo}(V \otimes \mathbb{C})_0$ with values in $\Lambda(\mathfrak{spo}(V)_1^*)$.

We fix a relatively compact open neighborhood \mathcal{X} of 0 in $\mathfrak{spo}(V)_0$. Since Spf is homogeneous of degree $\frac{n-m}{2}$ it follows that for any relatively compact open subset $\mathcal{W} \subset \mathfrak{spo}(V)_0$, there exists a constant $K_{\mathcal{W}}$ such that for any $(X, \mathbf{i}Y) \in \mathcal{W} \times \mathbf{i}(\mathcal{V}^- \cap \mathcal{X})$ and any homogeneous differential operator $\mathcal{D} \in \Lambda(\mathfrak{spo}(V)_1)$ we have for some $k \in \mathbb{N}$:

$$(127) \quad \left| (\mathcal{D} \text{Spf}_{an})(\mathbf{i}(X + \mathbf{i}Y)) \right| = \left| (\mathcal{D} \text{Spf}_{an})(-Y + \mathbf{i}X) \right| \leq K_{\mathcal{W}} N'(Y)^{-k}.$$

(N' is a norm on $\mathfrak{spo}(V)_0$.)

Then [Hör83, Theorem 3.1.15] shows that its limit when Y goes to 0 in \mathcal{V}^- exists as a generalized function on $\mathfrak{spo}(V)_0$ with values in $\Lambda(\mathfrak{spo}(V)_1^*)$. We have:

$$(128) \quad \text{Spf}_{gene}(X) = \lim_{Y \rightarrow 0, Y \in \mathcal{V}^-} \mathbf{i}^{\frac{m-n}{2}} \text{Spf}_{an}(\mathbf{i}(X + \mathbf{i}Y)).$$

Since Spf_{an} is the holomorphic extension of $\text{Spf}_{an}|_{\mathcal{V}^+}$. It is entirely determined by $\text{Spf}_{an}|_{\mathcal{V}^+}$. On the other hand, since $\text{Spf}_{gene}(X)$ is the limit of $\mathbf{i}^{\frac{n-m}{2}} \text{Spf}_{an}(\mathbf{i}(X + \mathbf{i}Y))$, Spf_{gene} is determined by Spf_{an} and thus by $\text{Spf}_{an}|_{\mathcal{V}^+}$.

In particular it follows that Spf_{gene} possesses the properties of the sections 2.3-2.8.

Let \mathcal{P} be a near superalgebra and $X \in \mathfrak{spo}(V)_{\mathcal{P}}(\mathcal{V}^+)$, let $\epsilon > 0$, we have:

$$(129) \quad \text{Spf}_{an}((\epsilon + \mathbf{i})X) = (\mathbf{i} + \epsilon)^{\frac{n-m}{2}} \text{Spf}_{an}(X)$$

It follows, taking the limit of (129) when ϵ goes to zero multiplied by $\mathbf{i}^{\frac{m-n}{2}}$:

$$(130) \quad \text{Spf}_{gene}|_{\mathcal{V}^+} = \text{Spf}_{an}|_{\mathcal{V}^+}.$$

($\phi|_{\mathcal{V}^+}$ denotes the restriction of the (generalized) function ϕ to the open set \mathcal{V}^+ .) In particular, Spf_{gene} is analytic on \mathcal{V}^+ .

From now on Spf stands for Spf_{gene} , and for $(X, Y) \in \mathfrak{spo}(V \otimes \mathbb{C})_{\mathcal{P}}(\mathcal{V}^+ \times \mathbf{i}\mathfrak{spo}(V))$ $\text{Spf}(X + \mathbf{i}Y)$ stands for $\text{Spf}_{an}(X + \mathbf{i}Y)$.

3.3. Evaluation of Spf on $\mathcal{V}_{p,q}$. We recall that $\mathcal{U}_{p,q} \subset \mathfrak{sp}(V_0)$ denote the open set of $X \in \mathfrak{sp}(V_0)$ such that $v \mapsto B(v, Xv)$ is a quadratic form of signature (p, q) on V_0 and $\mathcal{V}_{p,q} = \mathcal{U}_{p,q} \times \mathfrak{so}(V_1)$.

Proposition 3.1. *Let \mathcal{P} be a near superalgebra. Let $(p, q) \in \mathbb{N}^2$ such that $p + q = m$. Let $X \in \mathfrak{spo}(V)_{\mathcal{P}}(\mathcal{V}_{p,q})$. It means that X is represented in a symplectic basis $(e_1, \dots, e_m, f_1, \dots, f_n)$ by*

$$(131) \quad X = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \mathfrak{spo}(V)_{\mathcal{P}}, \text{ with } \mathbf{b}(A) \in \mathcal{U}_{p,q}$$

We have:

$$(132) \quad \text{Spf} \begin{pmatrix} A & B \\ C & D \end{pmatrix} = i^q \frac{\text{Pfaff}(D - CA^{-1}B)}{\sqrt{|\det(A)|}}.$$

where we recall that Pfaff is the ordinary Pfaffian.

Proof. The first equality in formula (107) implies that it is enough to prove the formula for $X \in (\mathfrak{spo}(\mathcal{V}_{p,q})_0)_{\mathcal{P}}$.

First, consider the particular case where $X \in \mathfrak{spo}(V)_{\mathcal{P}}(\mathcal{V}^+)$. In this case $(p, q) = (m, 0)$, thus the coefficient is $i^0 = 1$ and for $X \in \mathfrak{spo}(V)_{\mathcal{P}}(\mathcal{V}^+)$, $\mathbf{b}(A) \in \mathcal{U}^+$ and thus $\det(\mathbf{b}(A)) > 0$. The proposition reduces in this case to formula (107).

Since $\mathcal{V}_{p,q}$ is a purely even real manifold, it is enough to consider $\mathcal{P} = \mathbb{R}$ and $X \in \mathcal{V}_{p,q}$. We put $X = \begin{pmatrix} A & 0 \\ 0 & D \end{pmatrix}$, with $A \in \mathcal{U}_{p,q}$ and $D \in \mathfrak{so}(V_0)$.

Then:

$$(133) \quad \text{Spf}(X) = i^{\frac{m-n}{2}} \int_{V_0} d_{V_0}(v_0) \exp(-\frac{i}{2}B(v_0, Av_0)) \int_{V_1} d_{V_1}(v_1) \exp(-\frac{i}{2}B(v_1, Dv_1)).$$

On one hand:

$$(134) \quad \int_{V_1} d_{V_1}(v_1) \exp(-\frac{i}{2}B(v_1, Dv_1)) = i^{\frac{n}{2}} \text{Pfaff}(D).$$

On the other hand, it is well known (cf. for example [Hör83, formula 3.4.6]) that for $A \in \mathcal{U}_{p,q}$:

$$(135) \quad \int_{V_0} |dx^1 \dots dx^m| \exp(-\frac{i}{2}B(v_0, Av_0)) = \frac{(2\pi)^{\frac{m}{2}}}{\exp(i\frac{p-q}{4}\pi) \sqrt{|\det(A)|}}.$$

Since $p + q = m$, $\frac{i^{\frac{m}{2}}}{\exp(i\frac{p-q}{4}\pi)} = i^q$ and the formula follows. \square

In particular, it implies that Spf is smooth on \mathcal{V} (in fact it is analytic) and that for any $X \in \mathfrak{spo}(V)_{\mathcal{P}}(\mathcal{V})$, we have:

$$(136) \quad \text{Spf}(X)^2 = \text{Ber}^-(X).$$

3.4. Example: $\mathfrak{spo}(2, 2)$. Let us consider as an example the case $\mathfrak{g} = \mathfrak{spo}(2, 2)$. Let \mathcal{P} be a near superalgebra. The algebra $\mathfrak{spo}(2, 2)_{\mathcal{P}}$ is the set of matrices:

$$(137) \quad X = \left(\begin{array}{c|c} A & B \\ \hline C & D \end{array} \right) = \left(\begin{array}{cc|cc} a & b & \beta & \delta \\ c & -a & -\alpha & -\gamma \\ \hline -\alpha & -\beta & 0 & -d \\ -\gamma & -\delta & d & 0 \end{array} \right)$$

where $a, b, c, d \in \mathcal{P}_0$ and $\alpha, \beta, \gamma, \delta \in \mathcal{P}_1$. Here $V = \mathbb{R}^{(2,2)}$ is endowed with the symplectic form \mathbf{B} given in the canonical base (e_1, e_2, f_1, f_2) ($|e_i| = 0$ and $|f_i| = 1$) by $\mathbf{B}(f_i, f_j) = \delta_i^j$, $\mathbf{B}(e_1, e_2) = -\mathbf{B}(e_2, e_1) = 1$, $\mathbf{B}(e_1, e_1) = \mathbf{B}(e_2, e_2) = 0$ and $\mathbf{B}(e_i, f_j) = \mathbf{B}(f_j, e_i) = 0$, and the orientation defined by the basis (e_1, e_2) .

We have:

$$(138) \quad \begin{aligned} CA^{-1}B &= \frac{-1}{a^2 + bc} \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \begin{pmatrix} a & b \\ c & -a \end{pmatrix} \begin{pmatrix} \beta & \delta \\ -\alpha & -\gamma \end{pmatrix} \\ &= \frac{1}{a^2 + bc} \begin{pmatrix} 0 & -(a\alpha\delta - b\alpha\gamma + c\beta\delta + a\beta\gamma) \\ a\alpha\delta - b\alpha\gamma + c\beta\delta + a\beta\gamma & 0 \end{pmatrix}. \end{aligned}$$

We denote by Spf_0 the superpfaffian on $\mathfrak{sp}(\mathbb{R}^2)$. Thus, for $X \in \mathfrak{spo}(2, 2)_P(\mathcal{V})$ ($\mathcal{V} = \mathcal{U} \times \mathfrak{so}(2)$), we get from (132):

$$(139) \quad \text{Spf}(X) = \left(d - \frac{a(\alpha\delta + \beta\gamma) - b\alpha\gamma + c\beta\delta}{a^2 + bc} \right) \text{Spf}_0 \begin{pmatrix} a & b \\ c & -a \end{pmatrix}.$$

With (cf. formula (132)):

$$(140) \quad \text{Spf}_0 \begin{pmatrix} a & b \\ c & -a \end{pmatrix} = \frac{1}{\sqrt{-(a^2 + bc)}} \quad \text{if } \begin{pmatrix} a & b \\ c & -a \end{pmatrix} \in \mathcal{U}_{2,0}$$

$$(141) \quad \text{Spf}_0 \begin{pmatrix} a & b \\ c & -a \end{pmatrix} = \frac{-1}{\sqrt{-(a^2 + bc)}} \quad \text{if } \begin{pmatrix} a & b \\ c & -a \end{pmatrix} \in U_{0,2}$$

$$(142) \quad \text{Spf}_0 \begin{pmatrix} a & b \\ c & -a \end{pmatrix} = \frac{i}{\sqrt{a^2 + bc}} \quad \text{if } \begin{pmatrix} a & b \\ c & -a \end{pmatrix} \in \mathcal{U}_{1,1}.$$

Now we give formula (176) in this particular case. We denote by (x, y, ξ, η) the system of coordinates on $\mathbb{R}^{(2,2)}$ dual of (e_1, e_2, f_1, f_2) . Let $v = e_1x + e_2y + f_1\xi + f_2\eta$ be the generic point of $\mathbb{R}^{(2,2)}$. We have:

$$(143) \quad \mu(X, v) = d\eta\xi + \alpha\xi x + \beta\xi y + \gamma\eta x + \delta\eta y - \frac{c}{2}x^2 + axy + \frac{b}{2}y^2.$$

Thus, since $d_V(v) = \frac{1}{2\pi} |dx dy| \frac{\partial}{\partial \xi} \frac{\partial}{\partial \eta}$:

$$(144) \quad \int_{\mathbb{R}^{(2,2)}} d_V(v) \exp(i\mu(X, v)) = \frac{1}{2\pi} \int_{\mathbb{R}^2} |dx dy| \left(id - (\alpha x + \beta y)(\gamma x + \delta y) \right) \exp \left(i \left(-\frac{c}{2}x^2 + axy + \frac{b}{2}y^2 \right) \right),$$

Now we compute the integral on the right hand side. We put

$$(145) \quad \mathcal{H} = \begin{pmatrix} \frac{\alpha\delta + \beta\gamma}{2} & \beta\delta \\ -\alpha\gamma & -\frac{\alpha\delta + \beta\gamma}{2} \end{pmatrix}.$$

With this notation we have if $v_0 = e_1x + e_2y$ is the generic point of \mathbb{R}^2 :

$$(146) \quad -(\alpha x + \beta y)(\gamma x + \delta y) = \mathbf{B}(v_0, \mathcal{H}v_0)$$

Thus

$$(147) \quad \begin{aligned} & - \int_{\mathbb{R}^2} |dx dy| (\alpha x + \beta y)(\gamma x + \delta y) \exp \left(i \left(-\frac{c}{2}x^2 + axy + \frac{b}{2}y^2 \right) \right) \\ &= \int_{\mathbb{R}^2} |dx dy| B(v_0, \mathcal{H}v_0) \exp -\frac{i}{2} \mathbf{B}(v_0, Av_0) \\ &= 2i\partial_{\mathcal{H}} \int_{\mathbb{R}^2} |dx dy| \exp -\frac{i}{2} \mathbf{B}(v_0, Av_0). \end{aligned}$$

Finally, since

$$(148) \quad \text{Spf}_0 \begin{pmatrix} a & b \\ c & -a \end{pmatrix} = \frac{i}{2\pi} \int_{\mathbb{R}^2} |dx dy| \exp -\frac{i}{2} B(v_0, Av_0),$$

we have in $\mathcal{C}_{\mathfrak{spo}(V)}^{-\infty}(\mathfrak{spo}(V)_0)$:

$$(149) \quad \text{Spf}(X) = \left(d + 2 \left(\frac{\alpha\delta + \beta\gamma}{2} \frac{\partial}{\partial a} + \beta\delta \frac{\partial}{\partial b} - \alpha\gamma \frac{\partial}{\partial c} \right) \right) \text{Spf}_0 \begin{pmatrix} a & b \\ c & -a \end{pmatrix}.$$

From this equation we deduce again (139).

3.5. Singularities and wave front set. Let $\text{singsupp}(\text{Spf})$ be the set of singularities of the superPfaffian.

Lemma 3.2. *We have*

$$(150) \quad \text{singsupp}(\text{Spf}) = \mathfrak{spo}(V)_0 \setminus \mathcal{V}.$$

Proof. Proposition 3.1 implies that Spf is analytic on \mathcal{V} . Moreover, if $X \in \mathfrak{spo}(V)_0 \setminus \mathcal{V}$ and \mathcal{W} is a neighborhood of X , Spf is not bounded on $\mathcal{W} \cap \mathcal{V}$. Hence, X is a singularity. \square

Let ϕ be a generalized function on $\mathfrak{spo}(V)$. We denote by $\Sigma(\phi)$ the cone in $\mathfrak{spo}(V)^*$ defined by the following (cf. [Hör83, Formula (8.1.1)]).

Let N be the norm on $\mathfrak{spo}(V)$ defined in section 3.1. Let $f \in \mathfrak{spo}(V)^*$ then (cf. [Hör83, Formula (8.1.1)]) $f \notin \Sigma(\phi)$ if and only if $f = 0$ or if for any $k \in \mathbb{N}$, there exists $C_k \in \mathbb{R}$ such that for any h in some conic neighborhood of f :

$$(151) \quad |\widehat{\phi}(h)| \leq \frac{C_k}{(1 + N(h))^k}$$

Let $X \in \text{singsupp}(\text{Spf})$. We denote by $WF_X(\text{Spf})$ the wave front set of Spf at X :

$$(152) \quad WF_X(\text{Spf}) = \bigcap_{\phi, \phi(X) \neq 0} \Sigma(\phi \text{Spf}),$$

where ϕ run in the set of compactly supported smooth distributions on $\mathfrak{spo}(V)_0$.

We recall that $\tilde{\mu} : V \mapsto \mathfrak{spo}(V)^*$, and $\tilde{\mu}(v)(X) = \mu(X, v)$, thus $\tilde{\mu}(V_0) \subset \mathfrak{spo}(V)_0^*$.

Proposition 3.2.

$$(153) \quad WF_X(\text{Spf}) \subset \tilde{\mu}(V_0) \setminus \{0\}.$$

Proof. It follows from formula (107) that the wave front set of Spf on $\mathfrak{spo}(V)$ at X is equal to the wave front set of Spf_0 on $\mathfrak{sp}(V_0)$ at X . (The later is a subset of $\mathfrak{sp}(V_0) \oplus \mathfrak{sp}(V_0)^*$ which is canonically embedded in $\mathfrak{spo}(V)_0 \oplus \mathfrak{spo}(V)_0^*$ by mean of the decomposition $\mathfrak{spo}(V)_0 = \mathfrak{sp}(V_0) \oplus \mathfrak{so}(V_1)$). Thus from now on we assume that $V = V_0$.

Let ϕ be a smooth compactly supported distribution on $\mathfrak{sp}(V)$. Then, ϕSpf is a compactly supported distribution on $\mathfrak{sp}(V)$.

Let us precise $\widehat{\phi \operatorname{Spf}}(h)$.

$$\begin{aligned}
 \widehat{\phi \operatorname{Spf}}(h) &= \int_{\mathfrak{sp}(V)} \phi(X) \operatorname{Spf}(X) \exp(-i h(X)) \\
 &= i^{\frac{m}{2}} \int_V d_V(v) \int_{\mathfrak{sp}(V)} \phi(X) \exp\left(-\frac{i}{2} B(v, Xv)\right) \exp(-i h(X)) \\
 &= i^{\frac{m}{2}} \int_V d_V(v) \int_{\mathfrak{sp}(V)} \phi(X) \exp(-i(h - \tilde{\mu}(v))(X)) \\
 &= i^{\frac{m}{2}} \int_V d_V(v) \widehat{\phi}(h - \tilde{\mu}(v))
 \end{aligned}
 \tag{154}$$

Since ϕ is smooth and compactly supported, $\widehat{\phi}$ is rapidly decreasing. Thus for any $k \in \mathbb{N}$, there is a constant K_k such that:

$$|\widehat{\phi}(h - \tilde{\mu}(v))| \leq \frac{K_k}{(1 + N(h - \tilde{\mu}(v)))^k}
 \tag{155}$$

For $h \in \mathfrak{sp}(V)^*$ we put

$$d(h) = \min_{v \in V} \{N(h - \tilde{\mu}(v))\}.
 \tag{156}$$

We have if $h \neq 0$:

$$d(h) = N(h) d\left(\frac{h}{N(h)}\right)
 \tag{157}$$

Moreover, since $\tilde{\mu}(V)$ is closed there is $v_h \in V$ such that $N(h - \tilde{\mu}(v_h)) = d(h)$ (this propriety determines v_h up to multiplication by ± 1). Then we have: $N(h - \tilde{\mu}(v)) \geq N(h - \tilde{\mu}(v_h)) = d(h)$ and thus $N(\tilde{\mu}(v_h) - \tilde{\mu}(v)) \leq N(h - \tilde{\mu}(v_h)) + N(h - \tilde{\mu}(v)) \leq 2N(h - \tilde{\mu}(v))$. It follows

$$\begin{aligned}
 (1 + N(h - \tilde{\mu}(v)))^2 &\geq (1 + d(h)) \left(1 + \frac{1}{2} N(\tilde{\mu}(v_h) - \tilde{\mu}(v))\right) \\
 &\geq \frac{1}{2} (1 + d(h)) (1 + N(\tilde{\mu}(v_h) - \tilde{\mu}(v)))
 \end{aligned}
 \tag{158}$$

Therefore:

$$|\widehat{\phi}(h - \tilde{\mu}(v))| \leq \frac{2^{\frac{k}{2}} K_k}{(1 + d(h))^{\frac{k}{2}} (1 + N(\tilde{\mu}(v_h) - \tilde{\mu}(v)))^{\frac{k}{2}}}
 \tag{159}$$

Thus:

$$\left| \widehat{\phi \operatorname{Spf}}(h) \right| \leq \frac{2^{\frac{k}{2}} K_k}{(1 + d(h))^{\frac{k}{2}}} \int_V \frac{d_V(v)}{(1 + N(\tilde{\mu}(v_h) - \tilde{\mu}(v)))^{\frac{k}{2}}}
 \tag{160}$$

We have:

$$\begin{aligned}
 1 + N(\tilde{\mu}(v_h) - \tilde{\mu}(v)) &\geq 1 + |N(\tilde{\mu}(v)) - N(\tilde{\mu}(v_h))| \\
 &\geq \begin{cases} 1 + \frac{1}{2} N(\tilde{\mu}(v)) & \text{if } N(\tilde{\mu}(v)) \geq 2N(\tilde{\mu}(v_h)), \\ 1 & \text{otherwise.} \end{cases}
 \end{aligned}
 \tag{161}$$

We recall from section 3.1 that we fixed some $J \in \mathcal{U}^+$ and put for $v \in V$ $\|v\|^2 = \frac{1}{2}B(v, Jv) = -\mu(J, v)$. Then, from formula (118) we obtain $\|v\| \leq \sqrt{N(\tilde{\mu}(v))N'(J)}$.

We recall that for $r > 0$, with $m = \dim(V)$:

$$(162) \quad \int_{\{v \in V, \|v\| \leq r\}} d_V(v) = \frac{1}{(2\pi)^{\frac{m}{2}}} \frac{\pi^{\frac{m}{2}} r^m}{\frac{m}{2}!}$$

We have for $k \geq m \geq 2$:

$$(163) \quad \begin{aligned} \int_V \frac{d_V(v)}{(1 + N(\tilde{\mu}(v_h) - \tilde{\mu}(v)))^{\frac{k}{2}}} &\leq \int_{\{v \in V, \|v\| \leq \sqrt{2N(\tilde{\mu}(v_h))N'(J)}\}} d_V(v) \\ &\quad + \int_V d_V(v) \frac{1}{(1 + \frac{1}{2}N(\tilde{\mu}(v)))^{\frac{k}{2}}} \\ &\leq \frac{(2N(\tilde{\mu}(v_h))N'(J))^{\frac{m}{2}}}{2^{\frac{m}{2}} \frac{m}{2}!} + \int_V d_V(v) \frac{1}{(1 + \frac{1}{2}N(\tilde{\mu}(v)))^{\frac{k}{2}}} \\ &\leq \frac{(N(\tilde{\mu}(v_h))N'(J))^{\frac{m}{2}}}{\frac{m}{2}!} + \int_V d_V(v) \frac{1}{(1 + \frac{1}{2N'(J)}\|v\|^2)^{\frac{k}{2}}} \end{aligned}$$

We put $M'_k = 2^{\frac{k}{2}} K_k \frac{N'(J)^{\frac{m}{2}}}{\frac{m}{2}!}$ and $M''_k = 2^{\frac{k}{2}} K_k \int_V d_V(v) \frac{1}{(1 + \frac{1}{2N'(J)}\|v\|^2)^{\frac{k}{2}}}$. Thus:

$$(164) \quad \left| \widehat{\phi \text{Spf}(h)} \right| \leq \frac{M'_k N(\tilde{\mu}(v_h))^{\frac{m}{2}} + M''_k}{(1 + d(h))^{\frac{k}{2}}}.$$

Let \mathcal{C} be a conic neighborhood of f . We put:

$$(165) \quad M_{\mathcal{C}} = \min \{d(h) / h \in \mathcal{C} \text{ and } N(h) = 1\}.$$

Now assume that $f \notin \tilde{\mu}(V) \setminus \{0\}$ and $f \neq 0$ ($0 \notin WF_X(\text{Spf})$ by definition). Since $\tilde{\mu}(V)$ is closed, there is a conic neighborhood \mathcal{C} of f whose closure do not intersect $\tilde{\mu}(V) \setminus \{0\}$. In this case, $M_{\mathcal{C}} > 0$. It follows that for any $h \in \mathcal{C}$, $h \neq 0$:

$$(166) \quad N(\tilde{\mu}(v_h)) \leq d(h) + N(h) = d(h) \left(1 + \frac{1}{d(\frac{h}{N(h)})}\right) \leq (1 + d(h)) \left(1 + \frac{1}{M_{\mathcal{C}}}\right)$$

Thus

$$(167) \quad \begin{aligned} \left| \widehat{\phi \text{Spf}(h)} \right| &\leq \frac{M'_k (1 + d(h))^{\frac{m}{2}} \left(1 + \frac{1}{M_{\mathcal{C}}}\right)^{\frac{m}{2}} + M''_k}{(1 + d(h))^{\frac{k}{2}}} \\ &\leq \frac{M'_k \left(1 + \frac{1}{M_{\mathcal{C}}}\right)^{\frac{m}{2}} + M''_k}{(1 + d(h))^{\frac{k-m}{2}}}. \end{aligned}$$

We put $M_k = M'_k \left(1 + \frac{1}{M_{\mathcal{C}}}\right)^{\frac{m}{2}} + M''_k$. It follows for any $h \in \mathcal{C}$:

$$(168) \quad \left| \widehat{\phi \text{Spf}(h)} \right| \leq \frac{M_k}{(1 + M_{\mathcal{C}} N(h))^{\frac{k-m}{2}}}.$$

This proves that $f \notin WF_X(\text{Spf})$.

□

3.6. Uniqueness results. We put:

$$(169) \quad (\mathcal{V}^-)^0 = \{f \in \mathfrak{spo}(V)_0^* / \forall X \in \mathcal{V}^-, f(X) \geq 0\}.$$

We have $\tilde{\mu}(V_0) \subset (\mathcal{V}^-)^0$.

As a direct application of [Hör83, Theorem 8.4.15] we obtain.

Theorem 3.1. *Let V be a symplectic supervector space.*

Let \mathcal{V}^+ , \mathcal{V}^- , $(\mathcal{V}^-)^0$, $\text{singsupp}(\text{Spf})$, be defined as above.

Let $\phi \in \mathcal{C}_{\mathfrak{spo}(V)}^{-\infty}(\mathfrak{spo}(V)_0)$ be a generalized function on $\mathfrak{spo}(V)$, such that:

- (1) ϕ is smooth on \mathcal{V}^+ and $(\phi|_{\mathcal{V}^+})^2 = \text{Ber}^-$
- (2) $WF(\phi) \subset \mathfrak{spo}(V)_0 \times (\mathcal{V}^-)^0$.

Then, $\phi = \text{Spf}$ or $-\text{Spf}$. More precisely, an orientation of V_1 choose between Spf and $-\text{Spf}$.

Proof. Let $\phi \in \mathcal{C}_{\mathfrak{spo}(V)}^{-\infty}(\mathfrak{spo}(V)_0)$ satisfying the above conditions. Condition (2) and [Hör83, Theorem 8.4.15] imply that for any open convex cone Γ with closure included in $\mathcal{V}^- \cup \{0\}$, there is an analytic function F on $\mathfrak{spo}(V) \times \mathbf{i}\Gamma$ such that on $\mathfrak{spo}(V)$:

$$(170) \quad \phi(X) = \lim_{Y \rightarrow 0, Y \in \Gamma} F(X + \mathbf{i}Y).$$

We choose an orientation of V_1 . Condition (1) implies that on \mathcal{V}^+ , $\phi(X) = \pm \text{Spf}(X)$. Thus, by [Hör83, Theorem 3.1.15 and Remark], we have on $\mathfrak{spo}(V) \times \mathbf{i}\Gamma$

$$(171) \quad F(X + \mathbf{i}Y) = \pm \mathbf{i}^{\frac{m-n}{2}} \text{Spf}(\mathbf{i}(X + \mathbf{i}Y)).$$

where we recall from section 2 that $\text{Spf}(\mathbf{i}(X + \mathbf{i}Y))$ is the analytic function on $\mathcal{V}^+ \times \mathbf{i}\mathfrak{spo}(V)$ defined by formula (31). Thus equation (128) implies that $\phi = \pm \text{Spf}$.

Since we saw that changing the orientation of V_1 changes Spf to $-\text{Spf}$, the last remark follows. □

Remark: In the definition of Spf besides the orientation of V_1 , we chose a square root \mathbf{i} of -1 . Changing \mathbf{i} into $-\mathbf{i}$ changes $\text{Spf}(X)$ into $\overline{\text{Spf}}(X)$ which is the limit of $(-\mathbf{i})^{\frac{m-n}{2}} \text{Spf}(-\mathbf{i}(X + \mathbf{i}Y))$ when Y goes to 0 in $\mathcal{V}^+ = -\mathcal{V}^-$. In particular:

$$(172) \quad WF(\overline{\text{Spf}}) = -WF(\text{Spf}).$$

3.7. Linear subspaces and Subalgebras. Let \mathfrak{g} be a linear subspace of $\mathfrak{spo}(V)$. We put:

$$(173) \quad \mathfrak{g}_0^\perp = \{f \in \mathfrak{spo}(V)_0^* / f(\mathfrak{g}_0) = \{0\}\}$$

Let us assume that $\mathfrak{g}_0^\perp \cap \tilde{\mu}(V_0) = \{0\}$. This means:

$$(174) \quad \forall u \in V_0 \setminus \{0\}, \exists X \in \mathfrak{g}_0, \mu(X, u) = -\frac{1}{2}B(u, Xu) \neq 0.$$

In particular this condition is implied by the following:

$$(175) \quad \mathfrak{g}_0 \cap \mathcal{V}^- \neq \emptyset.$$

Indeed, assume that $\mathfrak{g}_0 \cap \mathcal{V}^- \neq \emptyset$, then there exists $X \in \mathfrak{g}_0$ such that $u \mapsto B(u, Xu)$ is negative definite on V_0 . in particular for any $u \in V_0 \setminus \{0\}$, $\mu(X, u) > 0$ and $\tilde{\mu}(u) \notin \mathfrak{g}_0^\perp$.

When this condition is realized, by standard results on generalized functions (cf. for example [Hör83, Corollary 8.2.7]), we can define $\text{Spf}|_{\mathfrak{g}}$. It is defined for a smooth compactly supported distribution t on \mathfrak{g} by:

$$(176) \quad \int_{\mathfrak{g}} t(X) \text{Spf}(X) = i^{\frac{m-n}{2}} \int_V d_V(v) \int_{\mathfrak{g}} t(X) \exp\left(-\frac{i}{2}B(v, Xv)\right).$$

Let us show that the preceding formula is meaningful.

Again, we take the notations of section 3.1. Since \mathfrak{g}_0 is a subalgebra of $\mathfrak{spo}(V)_0$ and $\mathfrak{sp}(V_0)$ and $\mathfrak{so}(V_1)$ are simple we have $\mathfrak{g}_0 = (\mathfrak{g}_0 \cap \mathfrak{sp}(V_0)) \oplus (\mathfrak{g}_0 \cap \mathfrak{so}(V_1))$. Here ρ will be a smooth compactly supported distribution on $\mathfrak{g}_0 \cap \mathfrak{sp}(V_0)$ and $X'' \in ((\mathfrak{g}_0 \cap \mathfrak{so}(V_1)) \oplus \mathfrak{g}_1)_{\mathcal{P}}$. We have $(v$ (resp. v_0)) is the generic point of V (resp. V_0):

$$(177) \quad \phi(X'', v) = \widehat{\rho}\left(-\widetilde{\mu}(v_0)|_{\mathfrak{g}_0 \cap \mathfrak{sp}(V_0)}\right) \exp\left(-\frac{i}{2}B(v, X''v)\right).$$

We put:

$$(178) \quad p = \min\{N(\widetilde{\mu}(u)|_{\mathfrak{g}_0 \cap \mathfrak{sp}(V_0)}) / u \in V_0, N(\widetilde{\mu}(u)) = 1\}.$$

Formula (118) gives for $v \in V_0$:

$$(179) \quad N(\widetilde{\mu}(u)|_{\mathfrak{g}_0 \cap \mathfrak{sp}(V_0)}) \geq pN(\widetilde{\mu}(u)) \geq \frac{p}{N'(J)}\|u\|^2.$$

The hypothesis implies that for $u \in V_0 \setminus \{0\}$, $N(\widetilde{\mu}(u)|_{\mathfrak{g}_0 \cap \mathfrak{sp}(V_0)}) > 0$. Thus, since $\{\widetilde{\mu}(u)/u \in V_0, N(\widetilde{\mu}(u)) = 1\}$ is compact, we have $p > 0$. Hence, since $\widehat{\rho}$ is rapidly decreasing on \mathfrak{g}^* , $\widehat{\rho}(-\widetilde{\mu}(v_0)|_{\mathfrak{g}_0 \cap \mathfrak{sp}(V_0)})$ is rapidly decreasing on V_0 and thus formula (176) is meaningful.

3.7.1. Symplectic 2-dimensional vector spaces. Let V be a Symplectic 2-dimensional vector space. Let $\mathfrak{g} \subset \mathfrak{spo}(V) \simeq \mathfrak{sl}(2)$ be a subalgebra. Then, $\mathfrak{g} = \mathfrak{g}_0$ and the condition $\mathfrak{g}_0^\perp \cap \widetilde{\mu}(V) \setminus \{0\} = \emptyset$ implies that $\mathfrak{g} = \mathfrak{sl}(2)$ or \mathfrak{g} is a compact Cartan subalgebra.

3.8. The case $V = V_0$: the superPfaffian as Fourier transform of a coadjoint orbit. In this section we assume that $V = V_0$. We recall from formula (115) that we denote by $\widetilde{\mu}$ the moment map from V to $\mathfrak{sp}(V)^*$; for any $v \in V$ and any $X \in \mathfrak{sp}(V)^*$:

$$(180) \quad \widetilde{\mu}(v)(X) = -\frac{1}{2}B(v, Xv).$$

We denote by $\widetilde{\mu}(V)$ the image of V by $\widetilde{\mu}$ in $\mathfrak{sp}(V)^*$. It is the disjoint union of a nilpotent orbit Ω and of $\{0\}$.

Let $\Omega = \widetilde{\mu}(V) \setminus \{0\}$. It is naturally endowed with an Hamiltonian structure. We denote by ω_Ω its symplectic form and by μ_Ω its moment map which is the identity map from $\mathfrak{sp}(V)$ onto $\mathfrak{sp}(V) \simeq (\mathfrak{sp}(V)^*)^* \subset C^\infty(\Omega)$. The form $\omega_\Omega^{\frac{m}{2}}$ determines an orientation on Ω . By definition the Fourier transform of Ω is the generalized function on $\mathfrak{sp}(V)$ defined by:

$$(181) \quad \mathcal{F}_\Omega(X) = \frac{1}{(2\pi)^{\frac{m}{2}}} \int_\Omega \exp(i(\mu_\Omega(X) + \omega_\Omega)).$$

Let us consider $\tilde{\mu}$ as a morphism of manifolds from $V \setminus \{0\}$ to Ω . Since B is an antisymmetric bilinear form on V , it is an element of $\Lambda(V^*)$ and thus determines a differential form ω_B of degree 2 on V . The Liouville integral of section 1.10 satisfies:

$$(182) \quad d_V(v) = \frac{1}{(2\pi)^{\frac{m}{2}} \frac{m}{2}!} |\omega_B^{\frac{m}{2}}(v)|.$$

Moreover the form $\omega_B^{\frac{m}{2}}$ fix an orientation of V . Since μ induces a morphism of Poisson algebras $\tilde{\mu} : S(V^*) \rightarrow \mathfrak{sp}(V) \subset \mathcal{C}^\infty(\Omega)$ (cf. (25)), we have

$$(183) \quad \tilde{\mu}^*(\mu_\Omega) = \tilde{\mu} \text{ and } \tilde{\mu}^*(\omega_\Omega) = \omega_B.$$

Since $\tilde{\mu}(V \setminus \{0\})$ is a double cover of Ω , we have:

$$(184) \quad \begin{aligned} \text{Spf}(X) &= \mathbf{i}^{\frac{m}{2}} \int_V d_V(v) \exp(-\mathbf{i}\tilde{\mu}(X)(v)) \\ &= \frac{1}{(2\pi)^{\frac{m}{2}}} \int_{V \setminus \{0\}} \exp(\mathbf{i}(\tilde{\mu}(X)(v) + \omega_B(v))) \\ &= 2 \frac{1}{(2\pi)^{\frac{m}{2}}} \int_\Omega \exp(\mathbf{i}(\mu_\Omega(X) + \omega_\Omega)) \\ &= 2\mathcal{F}_\Omega(X). \end{aligned}$$

3.8.1. *Example: Symplectic 2-dimensional vector space.* We take the same notations as in section 3.1.1.

We identify $X \in \mathfrak{sp}(V) \simeq \mathfrak{sl}(2)$ with its matrix in the basis (e_1, e_2) . We consider the restriction of Spf to the compact cartan subalgebra:

$$(185) \quad \mathfrak{t} = \left\{ \begin{pmatrix} 0 & -c \\ c & 0 \end{pmatrix} / c \in \mathbb{R} \right\}.$$

As a generalized function on \mathfrak{t} we have:

$$(186) \quad \text{Spf} \begin{pmatrix} 0 & -c \\ c & 0 \end{pmatrix} = \lim_{\epsilon \rightarrow 0, \epsilon > 0} \frac{1}{c + \mathbf{i}\epsilon}$$

On the other hand, Let $\alpha \in \mathfrak{t}^*$ such that:

$$(187) \quad \alpha \begin{pmatrix} 0 & -c \\ c & 0 \end{pmatrix} = 2\mathbf{i}c.$$

Then $(\alpha, -\alpha)$ is the root system of $(\mathfrak{sl}(2) \otimes \mathbb{C}, \mathfrak{t} \otimes \mathbb{C})$. Let Ω be the nilpotent coadjoint orbit of $\mathfrak{sl}(2)$ defined by

$$\Omega = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathfrak{sl}(2) / a^2 + bc = 0, c > 0 \right\}.$$

Then

$$\tilde{\mu}(V) = \Omega \cup \{0\}.$$

As it is well known, the restriction of the Fourier transform of Ω to regular elements $H = \begin{pmatrix} 0 & -c \\ c & 0 \end{pmatrix} \in \mathfrak{t}$ is:

$$(188) \quad \mathcal{F}_\Omega(H) = \int_\Omega \exp(\mathbf{i}\mu_\Omega(H) + \omega_\Omega) = \frac{\mathbf{i}}{\alpha(H)}.$$

It follows that:

$$(189) \quad \mathrm{Spf} \begin{pmatrix} 0 & -c \\ c & 0 \end{pmatrix} = 2\mathcal{F}_\Omega \begin{pmatrix} 0 & -c \\ c & 0 \end{pmatrix}.$$

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